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convention for  $\omega$ . Replace  
 $\omega$  by  $-\omega$  throughout to be  
consistent with later papers.*

MACH'S PRINCIPLE

and

A VARYING GRAVITATIONAL CONSTANT

by

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## ABSTRACT

In Part I it is conjectured that a "Mach's principle" might lead to a dependence of the local Newtonian gravitational constant,  $K$ , on universe structure,  $K \sim \frac{M}{R}$ . Einstein and others have suggested that general relativity predicts such a result. A closer analysis, however, including the carrying out of the geodesic equations to second order, seems to indicate that this is not true and that the apparent "Mach's principle" terms involving total universe structure are really only coordinate effects. Further, the measure of gravitating mass obtained in a local, proper Newtonian gravitational experiment is compared in a coordinate free way to an experimentally measurable inertial mass and found to be related to it in a way independent of the rest of the universe. A generalization of these results is given. It is based on the fact that in general relativity the only way the universe can influence experiments done in an electrically shielded lab is through the metric and that this can be "transformed away" to any degree of accuracy for a sufficiently small lab. Consequences of this are summarized in Dicke's "strong principle of equivalence." It is noted, however, that there are other statements which might be called "Mach's principles" which are satisfied in general relativity.

Part II is mainly concerned with the introduction of a varying gravitational constant into the framework of general relativity, violating the strong while preserving the weak principle of equivalence (i. e. geodesics for uncharged test particles). To this end a scalar field,  $\phi$  roughly corresponding to  $K^{-1}$  is added to the variational principle of general relativity. A weak field analysis of the resulting field equations not only yields the required Newtonian limit but also suggests contributions of local matter to  $\phi$  consistent with  $K^{-1} \sim \frac{M}{R}$ . These field equations are compared to Jordan's. The differences include not only the use of  $K^{-1}$  rather than  $K$  as a field variable but also our attempt here to relate  $\phi$  to a locally measured Newtonian constant. This requires the use of equations of motion correct at least through second order in  $\frac{K M}{R}$ . Infeld's method is used for this purpose. The result is that the theory does predict an influence on  $\phi$  consistent with  $K^{-1} \sim \frac{M}{R}$ . The analogue of the Schwarzschild solution is stated and the "three standard tests" evaluated from it. The Einstein results are approached for large absolute values of  $\omega$ , the coupling constant used for  $\phi$ . The same solution is obtained in isotropic coordinates where it can be written more simply. To completely determine  $\phi$  a boundary condition must be added. The condition  $\phi \rightarrow 0$  outside matter is proposed as it is noticed that

this would result in a "breakdown" of the field equations in the absence of matter. Lack of an exact interior solution hinders discussion of the results of such a condition. However, it is shown that for a fluid type mass shell universe it requires pressures of the order of densities. The cosmological problem is discussed in relation to Dirac's  $k \sim t$  cosmology. Again, no exact general solutions are known yet. However, the first few terms of an expansion in terms of  $\omega^{-1}$  fail to yield such a result. Finally, conservation laws are discussed and conserved total "energies" as well as total "gravitating radii" are exhibited.

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## PART I

### I. Introduction

#### A. Mach's Principle

The principle idea which guided Einstein in formulating the general theory of relativity was the local equivalence of gravitational and inertial effects, that is, the equivalence of a uniform gravitational force field and a constant acceleration of the reference frame. Another idea relating gravity and inertia is Mach's principle. This is less precisely formulated but suggests that the inertial properties of a body are determined by the distribution of matter in the universe. Since the gravitational field interacts with all matter, one could hope to see the Mach principle relationship between inertia and distant matter described in terms of the gravitational field. To state this in a way independent of units, consider the ratio of the inertial mass of a body to its active gravitational mass.

#### B. Gravitational Constant from Dimensional Analysis

In particular, let us see what this ratio might be in a static universe consisting only of a mass shell of radius  $R$  and inertial mass  $M$  together with a relatively small body of inertial mass  $m$  at its center. If we probe the gravitational field of  $m$  with a small test particle, we might expect from the Eötvös experiment that the acceleration of the test particle is independent of its mass. It certainly depends, however, on  $m$  and  $r$  and conceivably on  $M$  and  $R$ . The fact that the Newtonian theory of gravity is valid to a high degree of accuracy suggests that the

acceleration is

$$1) \quad a = - \frac{m}{\lambda^2 F(M, R)}$$

where  $F$  is a function of dimensions  $ML^{-1}$  (velocity of light  $c = 1$ .)

Dimensional analysis then suggests

$$2) \quad F = M/R$$

For a more general type of universe with masses  $m_a$  at distance  $r_a$  from some point  $x$ , this might be extended to

$$3) \quad F(x) = \sum_a \frac{m_a}{r_a}$$

Until recently experimental determinations of  $F$  from 1) were possible only on the earth. The value found is not inconsistent with 3) and present astronomical knowledge of  $m_a$ ,  $r_a$ . It is clear that in a uniform universe,  $m_a \sim r_a^2$  so that the dominant contribution in 3) comes from distant matter and is fairly constant in space and time. This also is consistent with present observations.

To determine an active gravitational mass,  $m_g$ , from 1) it is convenient to multiply and divide the right side by  $K_0/8\pi$ , a constant number of dimensions  $LM^{-1}$  and equal to the presently observed terrestrial value of  $1/F$ . Thus

$$4) \quad a = - \frac{K_0}{8\pi\lambda^2} \left( \frac{8\pi m}{K_0 F} \right)$$

The quantity in parentheses has dimensions of mass and will be called the active gravitational mass of  $m$ . In other words, a Cavendish experiment, interpreted in the context of a Newtonian theory with fixed gravitational constant  $K_0$ , would give a measurement of active gravitational mass,  $m_g = \frac{8\pi m}{K_0 F}$ .

Thus

$$5) \quad m/mg = \sum_a \frac{K_0 m_a}{8\pi \Lambda a}$$

C. Einstein's Interpretation  
of Results in General Relativity

Einstein<sup>1</sup> claims to find such a result in general relativity. As pointed out above, the major contribution to 5) is from distant matter and nearly constant. Hence, 5) can be written near the earth as

$$6) \quad m/mg \cong 1 + \sum_{\substack{\text{local} \\ \text{matter}}} \frac{K_0 m_a}{8\pi \Lambda a}$$

For local matter, however, each  $\frac{K_0 m_a}{8\pi \Lambda a}$  is small compared to one. This suggests that a weak field approximation be used to check 6). Einstein does this and arrives at

$$7) \quad \left(1 + \sum_{\substack{\text{local} \\ m_a}} \frac{K_0 m_a}{8\pi \Lambda a}\right) a \cong - \frac{K_0 m}{8\pi \Lambda^2}$$

Thus

$$8) \quad m/mg \cong 1 + \sum_{\substack{\text{local} \\ m_a}} \frac{K_0 m_a}{8\pi \Lambda a}$$

which is identical with 6). Einstein argued from this that since local

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<sup>1</sup>A. Einstein, The Meaning of Relativity, Princeton, 1955, Fifth edition, p. 99-108.

matter contributed to the ratio,  $m/m_g$ , all the universe probably does (Section II). There has been some discussion<sup>1a</sup> of what the numerical co-efficient of  $\sum_{\substack{m_a \\ m_a}} \frac{K_a m_a}{8\pi r_a}$  in the right side of 8) should be and indeed, the first approximation procedure seems inadequate to resolve this. Consequently, the equations of motion through second order will be applied to this problem in Section II, pages 6f.

#### D. Coordinate Dependence of This Result and Corresponding Invariant Statement

This result 8) or its corrected form II-11) below is clearly coordinate dependent, however. Hence the relationship between its numerical description of the path of a particle and the actually observed path is not defined without further analysis. The usual interpretation of general relativity is based on the identification of the invariant theoretical measure of an interval, proper time, with time experimentally measured in some fundamental way, e. g., on an atomic clock. An invariant measure of distance and thus acceleration can be obtained from this by setting the velocity of light one. When this is done, the invariant description of the path of a test particle relative to a central mass is found to be approximately Newtonian with coefficients independent of the rest of the universe. (Section III, pages 10f.)

#### E. Relation of m to Inertial Mass

However, the number M appearing in 5) has not yet been related to an experimentally measured inertial mass. To remedy this, a description of a process for invariantly studying the acceleration of charged bodies in a known electric field is given. The resultant ratio of "force"

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<sup>1a</sup> W. Davidson, Monthly Notices Roy. Astron. Soc. 117, 212.

to acceleration is defined as the inertial mass. For a simple theory of matter  $m_{\text{inert}}$  is found to be just the  $m$  appearing in 5) (Section IV, p. 13f.) This procedure assumes a given standard of charge and time interval.

F. Independence of the Relationship  
Between Inertial and Gravitational Mass  
from the Rest of the Universe:  
Strong Principle of Equivalence

The independence of the relationship between the two numbers,  $m_g$  and  $m_{\text{inert}}$ , from the rest of the universe is more generally true than the above special case might indicate. In other words, if an electrically shielded empty lab is almost flat, then the introduction of small masses and charges within and the study, both theoretical and experimental, of their motions and interactions is independent of the rest of the universe. This is because once the lab is shielded the only way the rest of the universe could influence it, according to general relativity, is through the metric. If this is sensibly flat within, then there can be no influence within. This is Dicke's "strong principle of equivalence." (Section V, pages 26 f.)

G. Other "Mach's Principles"

There are, however, other statements which might be considered Mach's principles. Two of these are briefly discussed. First, the universe can change the inertial and gravitational mass of a body, e.g., by heating it, but as pointed out above it does not enter into a statement of their relationship. Second, the motion of inertial, i.e., almost flat, frames is determined by the mass distribution in the

universe. (Section V.)

II. Einstein's Interpretation of Approximate Relative  
Coordinate Acceleration of Two Masses  
Inside a Static Spherical Shell

A. Introduction: Weak Field Metric and Geodesic

Gravity and general relativity being largely concerned with the interaction between masses as masses, Einstein was naturally interested in whether or not a Mach's principle as discussed in Section I above was satisfied in general relativity. Specifically, is the attraction and resultant relative motion of two gravitating bodies influenced by the rest of the universe?

Einstein<sup>2</sup> investigated this in the weak field approximation. The metric he found to represent the gravitational field due to a distribution of small masses corresponding to a "density"  $\sigma$  and having small velocities,  $\frac{dx^i}{ds}$  can be written as

$$g_{00} = 1 - \frac{K}{4\pi} \int \frac{\sigma dV}{r}$$

$$1) \quad g_{0i} = \frac{K}{2\pi} \int \frac{\sigma \frac{dx^i}{ds} dV}{r}$$

$$g_{ij} = -\delta_{ij} \left( 1 + \frac{K}{4\pi} \int \frac{\sigma dV}{r} \right)$$

on replacing Einstein's imaginary time  $x^4$  by the real  $x^0 = -ix^4$ . Equation 1) is correct only to order 1 in  $K \int \frac{\sigma dV}{r}$ , and  $\frac{dx^i}{ds}$ . The geodesic equation for a test particle in this field becomes

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2. Ibid.

$$2) \quad \frac{d}{dx^0} \left[ (1 + \bar{\sigma}) \vec{v} \right] = \nabla \bar{\sigma} + \frac{\partial \vec{A}}{\partial x^0} + (\nabla \times \vec{A}) \times \vec{v}$$

$$\vec{v} \equiv \frac{d\vec{x}}{dx^0}$$

where

$$\bar{\sigma} \equiv \frac{K}{8\pi} \int \frac{\sigma}{r} dV$$

$$3) \quad \vec{A} \equiv \frac{K}{2\pi} \int \frac{\sigma \frac{d\vec{x}}{dx^0} dV}{r}$$

B. Static Shell Universe: Einstein's Interpretation;  
Restatement in Terms of Active Gravitational Mass  
and Gravitational Constant

For simplicity consider the application of these results to the case of the motion of a test particle near a small mass,  $m$ , at rest at the origin, all inside a static, spherical shell of mass  $M$  and radius  $R$ .<sup>3</sup> Here 2) becomes

$$4) \quad \frac{d}{dx^0} \left[ \left( 1 + \frac{KM}{8\pi R} + \frac{Km}{8\pi r} \right) v^i \right] = \frac{Km}{8\pi} \frac{\partial}{\partial x^i} \frac{1}{r}$$

Thus,  $\left[ 1 + \frac{K}{8\pi} \left( \frac{M}{R} + \frac{m}{r} \right) \right]$  times the coordinate acceleration of the test particle is just the Newtonian term, to this approximation.

Einstein interpreted this by saying that the "inert mass is proportional to  $1 + \sigma$ "<sup>4</sup>, or in 4) to  $\left[ 1 + \frac{K}{8\pi} \left( \frac{M}{R} + \frac{m}{r} \right) \right]$ . However, an

<sup>3</sup> This example, while admittedly rather specialized, is sufficient to illustrate the ideas under consideration.

<sup>4</sup> A. Einstein, *Ibid.*, p. 102.

equivalent statement, more convenient for this discussion and in keeping with that of Section I, can be made. Specifically, dividing 4) by  $\left[1 + \frac{K}{8\pi} \left(\frac{M}{R} + \frac{m}{\lambda}\right)\right]$  gives for  $v^i$  instantaneously zero,

$$5) \quad \frac{d}{dx^0} v^i = \frac{K m}{8\pi \left[1 + \frac{K}{8\pi} \left(\frac{M}{R} + \frac{m}{\lambda}\right)\right]} \frac{\partial}{\partial x^i} \frac{1}{\lambda}$$

This, in keeping with Einstein's interpretation above, would suggest that the locally measured Newtonian active gravitational mass of  $m$  is

$$6) \quad m_g = \frac{m}{1 + \frac{K}{8\pi} \left(\frac{M}{R} + \frac{m}{\lambda}\right)}$$

or that the locally measured Newtonian gravitational constant is

$$7) \quad K_E = \frac{K}{1 + \frac{K}{8\pi} \left(\frac{M}{R} + \frac{m}{\lambda}\right)}$$

If this is true, a comparison of 6) with I-5) would show that a Mach's principle in the sense of Section I would be satisfied in general relativity, since the number  $K_E$  in 7), measuring the attraction of  $m$  for test particles would depend on the mass distribution,  $M/R$ , in the rest of the universe.

### C. An Objection; Correction to Higher Order

One objection<sup>5</sup> that might be raised against the above procedure is based on the fact that 4) and thus 5) are true only to first order in  $\frac{KM}{R}$  and  $\frac{K m}{\lambda}$ . Hence the "Mach's principle" terms in 5) are of higher order than can be consistently be retained.

5 See also in this connection W. Davidson, Monthly Notices, Roy. Astron. Soc. 117, 212. Davidson criticizes Einstein's retention of the  $\overline{v^i v^j}$  term because of the assumed smallness of  $v^i$ . He "corrects" this by retaining all velocity terms in the geodesic equation. His result is still questionable, however, on the basis of the discussion following in the text.

In other words, the difference between 5) and

$$8) \quad \frac{d}{dx^0} v^i = \frac{km}{8\pi} \frac{\partial}{\partial x^i} \frac{1}{r}$$

is too small to be retained in view of the approximations made in deriving 5). Consequently, to the accuracy assumed, 6) should be written

$$9) \quad m_g = m$$

and

$$10) \quad K_E = K$$

This objection, however, can be overcome by studying the equations of motion to higher order.<sup>6</sup> The result, for the same type of universe is

$$11) \quad \frac{d}{dx^0} v^i = \frac{km}{8\pi \left(1 + \frac{5km}{8\pi R}\right)} \frac{\partial}{\partial x^i} \frac{1}{r}$$

In equation 11), terms of order  $\left(\frac{km}{r}\right)^2$  have been neglected as well as terms of order,  $\frac{\Delta}{R}$ . These are not relevant to this discussion and for a situation of physical interest would be small compared to the terms kept. Terms of order  $\left(\frac{KM}{R}\right)^2$  and  $\frac{KM}{R} \cdot \frac{km}{r}$  have not been neglected, however, so that equations analogous to 6) and 7)

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<sup>6</sup>A. Papapetrou, Proc. Phys. Soc. (London) 64A, 57 (1951),  
V. Fock, J. Phys. (U.S.S.R.) 1, 81 (1939), L. Infeld, Rev.  
Mod. Phys. 29, 398 (1957).

might be written

$$12) \quad m_g = \frac{m}{1 + \frac{5KM}{8\pi R}}$$

and

$$13) \quad K_E = \frac{K}{1 + \frac{5KM}{8\pi R}}$$

There is, however, another objection that might be raised against these results. This is discussed in the following section.

### III. Locally Measured Newtonian Gravitational Constant and Active Gravitational Mass in General Relativity

#### From Invariant, Proper Distance-Time-Acceleration Measurements

##### A. Introduction: Meaning of Coordinates and Their Relation to Measurements

Since II-11) contains coordinate acceleration and distance, there is a question associated with the interpretation of it in Section II. This question concerns the meaning of coordinates and the metric tensor. The usual interpretation of general relativity rests on the identification of

$$1) \quad d\tau \equiv \left( -g_{\mu\nu}(x) dx^\mu dx^\nu \right)^{\frac{1}{2}} \quad ; \quad (\text{if } d\tau^2 \geq 0)$$

as the differential of "proper time," or time read on some basic, e.g., atomic, clock associated with the coordinate interval  $dx^\mu$ . Defining

the velocity of light to be one, and assuming a light ray to be a null geodesic, provides the basis for a method of obtaining a "proper" measurement of a "distance" between particles. Specifically, the proper distance between two time like paths will be taken as one half the proper time of flight (measured along one path) of a light ray from one path to the other and back again.<sup>7</sup> This provides a coordinate free, if impractical, method for obtaining a measurable, numerical description of the relative motion of two bodies.

### B. Application to Local Gravitational Acceleration;

#### Conclusion

An application of this method to II-11) yields

$$2) \quad \frac{d^2}{d\chi_p^0} \chi_p^i = \frac{K m}{8\pi} \frac{\partial}{\partial \chi_p^i} \frac{1}{r_p}$$

where  $\chi_p^i$  is proper distance as measured from the test particle to  $m$  and  $\chi_p^0$  is proper time along the test particle. In obtaining 2) higher order terms in  $\frac{K m}{r}$  and  $\frac{\Delta}{R}$  were neglected but the  $\frac{K M}{R} \frac{K m}{r}$  term was kept and cancelled out.

Thus, a coordinate free description of the motion shows that it is independent of the mass distribution in the rest of the universe, at least to the order of approximation for which 2) is valid. Hence, this example does not seem to indicate the validity of a physically detectable Mach's principle in general relativity in the sense of Sections I and II.

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7 E. Wigner, Rev. Mod. Phys. 29, 255 (1957).

### C. Calculations Leading to III-2

The calculation leading from II-11) to 2) is straightforward. First of all, it should be noted that  $m$  must be replaced by a non-singular source as used in the Papapetrou-Fock method before a proper distance between its center and that of the test particle can be defined. However, for the purpose of the discussion above, terms of order  $(\frac{K_m}{\lambda})^2$  are neglected and, since both sides of II-11) are already of first order in  $\frac{K_m}{\lambda}$ , this means that contributions to the metric from  $m$  can be neglected. Hence, Infeld's renormalized delta function, which disregards self interaction could equally well be used.

Secondly, since II-11 is accurate only through terms,  $(\frac{KM}{R})^2$ ,  $\frac{KM}{R} \frac{K_m}{\lambda}$ , and both sides of II-11) are already of order  $\frac{K_m}{\lambda}$ , only terms linear in  $\frac{KM}{R}$  in the metric need be kept in converting the distances and times in II-11) to proper units. The fact that only the "first order" terms need be kept is important because the metric obtained by Infeld coincides to this order with that of Papapetrou-Fock. Further, the coordinate description of the motion of  $\mathcal{N}$  bodies through second order, of which II-11) is a special case, is the same in either method. Thus, the really observable prediction, a relation of proper relative accelerations to proper distances and velocities, is identical in both cases.

Finally, for the example at hand, all particles are instantaneously at rest and terms  $\frac{\lambda}{R}$  are to be neglected. This essentially means that between the test particle and  $m$ , changes in the background metric, i. e., neglecting  $m$ , can be ignored. Thus

$$3) \quad \begin{aligned} x_p^0 &\cong (-g_{00}(0))^{\frac{1}{2}} x^0 \\ x_p^i &\cong (-g_{ii}(0))^{\frac{1}{2}} x^i \end{aligned}$$

$$\bar{g}_{\alpha\beta} \equiv \text{background metric, i. e., with } m = 0$$

where  $x_p^0$  and  $x_p^i$  are proper time and distance as defined above.

For the example at hand

$$4) \quad \overline{g_{00}}(0) \approx -1 + \frac{2KM}{8\pi R}$$

$$\overline{g_{ij}}(0) \approx \delta_{ij} \left(1 + \frac{2KM}{8\pi R}\right)$$

Thus II-11) becomes

$$5) \quad \left(1 - \frac{2KM}{8\pi R}\right) \frac{d^2}{dx_p^0{}^2} \frac{x_p^i}{\left(1 + \frac{KM}{8\pi R}\right)} = \frac{Km \left(1 + \frac{2KM}{8\pi R}\right)}{8\pi \left(1 + \frac{5KM}{8\pi R}\right)} \frac{\partial}{\partial x_p^i} \frac{1}{r_p}$$

which immediately reduces to 2).

The question of what the "m" appearing in 2) means and how it is to be measured will now be taken up.

#### IV. Relationship Between Stress-Energy Tensor of Matter and Experimentally Observed Inertial and Active Gravitational Mass

##### A. Introduction

The discussion in Section III did not say anything about the physical meaning of the number m appearing in the right side of III-2). Mathematically this number m came from the stress tensor of matter in the Einstein equations and in fact, if  $W^A$  = four velocity of the particle, it is

$$1) \quad m = - \int_{x^0 = \text{constant}} W^0 T^0_{\alpha} \sqrt{-g} d^3x \quad \text{\textit{LABEL 4.1}}$$

for either the Infeld or the Papapetrou-Fock method (see eqn. 24, page 21; eqn. IX-6, page 70).

However, this is still not enough and merely replaces one unknown by another,  $m$  by  $T_{\alpha}^{\beta}$ , leaving the physical meaning and method of measurement of the latter undefined.

This section will consider one of two coordinate free methods for obtaining physically measurable numbers associated with the tensor  $T_{\alpha}^{\beta}$ , namely the measurement of inertial mass. The other, active gravitational mass was considered in sections II and III. Thus  $T_{\alpha}^{\beta}$  is considered as a mathematical intermediary between two observed numbers.

In order to measure inertial mass, "standard" electromagnetic theory will be assumed with its conserved current density. This gives a constant total charge and it is assumed that quantities of this charge are physically available in arbitrarily small amounts. A standard Coulomb law experiment, in proper units, will then provide a method for measuring inertial mass, the units of which will thus be determined solely in terms of a unit of time and charge.

The two numbers, inertial and active gravitational mass will be proved approximately equal for the case of a "fluid," irrespective of what the mass distribution in the rest of the universe is, provided, of course, that it does not encroach on the laboratory and that the latter and the masses and charges in it are sufficiently small.

As far as this paper is concerned, the main significance of this result is not so much the equality of active gravitational and inertial mass but the fact that the "function" expressing one in terms of the other is independent of the rest of the universe. A generalization of this result will be sketched in Section V, pages 26 f.

B. Electromagnetic Theory: Field Equations;  
Conservation of Charge; Equations of Motion

Standard electromagnetic theory in general relativity is based on the following equations

$$2) \quad F^{\alpha\beta}_{;\alpha} = \sigma W^{\beta} \quad ; \quad W^{\beta} W_{\beta} = -1$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

where  $\sigma$  is a scalar,  $F_{\alpha\beta}$  an antisymmetric tensor and  $W^{\beta}$  the four velocity of the charge. From 2) and the antisymmetry of  $F^{\alpha\beta}$  follows

$$3) \quad (\sigma W^{\beta})_{;\beta} = 0$$

or

$$4) \quad (\sqrt{-g} \sigma W^{\beta})_{;\beta} = 0$$

Hence, if on each surface  $x^0 = \text{constant}$ ,  $\sigma$  is zero outside a bounded region then

$$5) \quad q \equiv \int_{x^0=t} \sqrt{-g} \sigma W^0 d^3x$$

is independent of  $t$ . This constant number,  $q$ , is defined as the total charge of the distribution represented by  $\nabla W^{\alpha}$ . It is assumed that a unit of charge as defined by 5) is physically available. The operational definition of inertial mass as described later in this section

is fundamentally based on this association of the theoretical number 5) with a given, physical "charged" particle.

The equations of motion follow from conservation of the total stress-energy tensor given by

$$6) \quad T_{total}^{\alpha\beta} = T_m^{\alpha\beta} + \frac{1}{4} g^{\alpha\beta} F_{\lambda\mu} F^{\lambda\mu} - F^\alpha{}_\rho F^{\beta\rho}$$

Thus,  $(T_{total}^{\alpha\beta})_{;\beta} = 0$  becomes

$$7) \quad T_m^{\alpha\beta}{}_{;\beta} = \sigma w^\rho F^\alpha{}_\rho$$

### C. Papapetrou's Equations of Motion:

#### His Choice of Stress Tensor;

#### "Mass Density," Pressure; Definition of m

Papapetrou's<sup>8</sup> derivation of the equations of motion in general relativity is based on the conservation equations, which in the absence of charge become

$$8) \quad \int_m \alpha^{\beta}{}_{;\beta} - \frac{1}{2} \int_m g^{\mu\nu}{}_{;\alpha} = 0$$

where

$$9) \quad \int_m \alpha^{\beta} \equiv \sqrt{-g} T_m^{\alpha\beta}$$

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8 A. Papapetrou, Ibid.

The standard choice of  $\int_m \Gamma^{\alpha\beta}$  for a fluid is

$$10) \quad \int_m \Gamma^{\alpha\beta} = (\rho + P) W^\alpha W^\beta + P g^{\alpha\beta}$$

$$W^\alpha = \frac{dx^\alpha}{d\tau} \quad ; \quad d\tau \equiv (-g_{\alpha\beta} dx^\alpha dx^\beta)^{1/2}$$

with  $\rho$  and  $P$  scalars and  $P$  much smaller than  $\rho$ . Papapetrou only defines his choice of  $\int_m \gamma^{\alpha\beta}$  through the approximations necessary to derive the equation of motion through second order. He assumes that for a situation represented by  $n$  "particles" at points  $x_a^i$ ,  $\int_m \gamma^{\alpha\beta}$  is the sum of  $n$  terms,

$$11) \quad \int_m \gamma^{\alpha\beta} = \sum_{a=1}^n \int_a \gamma^{\alpha\beta}$$

with each  $\int_a \gamma^{\alpha\beta}$  vanishing outside a small region around  $x_a^i$  with the radius of this region much smaller than the separation  $|x_a^i - x_b^i|$  between any pair of particles. Changing to a metric of signature  $(-, +++)$  his choice for  $\int_a \gamma^{\alpha\beta}$  to the necessary order becomes

$$\int_a \gamma_0^0 = -\bar{\rho}_a \left( 1 + \frac{v_a^2}{2} - U_a + \frac{4u_a}{2} \right)$$

$$\int_a \gamma_0^k = -\bar{\rho}_a \dot{x}_a^k \left( 1 + \frac{v_a^2}{2} - U_a + \frac{4u_a}{2} \right) - \rho_a \dot{x}_a^{0k}$$

12)

$$\int_a \gamma_i^k = \bar{\rho}_a \dot{x}_a^i \dot{x}_a^k + \delta_i^k \rho_a$$

$$\int_a \gamma_i^0 = -\int_a \gamma_0^i + 4 \sum_b \bar{\rho}_b u_b (\dot{x}_a^i - \dot{x}_b^i)$$

where

$$13) \quad U_a \equiv U(x_a^i) \equiv \sum_b u_b(x_a^i)$$

$$\nabla^2 u_b(x^i) = -4\pi G \bar{\rho}_b$$

$$v_a^2 \equiv \dot{x}_a^i \dot{x}_a^k \delta_{ik}$$

He assumes that the functions  $\bar{\rho}_a(x)$  and  $P_a(x)$  are spherically symmetric about  $x_a^i$ . This is not an unreasonable assumption since "distorting" forces on  $a$  from gravitating mass  $M$  at a distance  $R$  would be of the order  $\frac{KM}{R^2} \frac{l_a}{R}$  where  $l_a$  is the distance across the  $a^{\text{th}}$  mass. From the assumption above,  $l_a \ll R$ , so that this "distorting" force can be neglected compared to the gravitational force  $\frac{KM}{R^2}$ . Finally, he also assumes that the velocities  $\dot{x}_a^i$  are small compared to one,  $v_a^2$  being of order two, the same order as  $U_a$ .

The mass,  $m$ , which appeared as the active gravitational mass in II-11) above is defined for the  $a^{\text{th}}$  particle by

$$14) \quad m_a \equiv \int_{R_a} \bar{\rho}_a d^3x$$

where  $R_a$  is a region containing all points at which  $\bar{\rho}_a \neq 0$  but none where  $\bar{\rho}_b \neq 0$  for  $b \neq a$ . He obtains,

$$15) \quad \frac{d}{dt} \bar{\rho}_a = 0 = \frac{d}{dt} m_a$$

and

$$16) \quad P_{a,i} = \bar{\rho}_a u_{a,i}$$

from the lowest order equations of motion. From 16) and the spherical symmetry of  $P_a$ ,  $\bar{P}_a$ , it follows that

$$17) \quad \int_{R_a} P_a d^3x = \frac{1}{6} \int_{R_a} \bar{P}_a u_a d^3x$$

The lowest order terms in the metric tensor can then be written

$$g_{00} = - (1 - 2U)$$

$$18) \quad g_{0i} = - 4 \sum_a u_a \bar{x}_a^i$$

$$g_{ik} = \delta_{ik} (1 + 2U)$$

$$\sqrt{-g} = 1 + 2U$$

#### D. Comparison of Papapetrou's Variables to Usual Choices; Meaning of his "m"

The comparison between Papapetrou's choice, 12), and the standard one in equation 10) now follows immediately. Inserting 18) into 10) and carrying out the operations to an order comparable to 12) gives

$$\begin{aligned}
 19) \quad \gamma_{a0}^0 &= -\rho_a (1 + 2U_a + v_a^2) \\
 \gamma_{a0}^k &= -\rho_a \dot{x}_a^k (1 + 2U_a + v_a^2) - \rho_a \dot{x}_a^k \\
 \gamma_{ai}^k &= \rho_a \dot{x}_a^i \dot{x}_a^k + \rho_a \delta_i^k \\
 \gamma_{ak}^0 &= -\gamma_{a0}^k + 4 \sum_D U_D \rho_a (\dot{x}_a^k - \dot{x}_b^k)
 \end{aligned}$$

Hence, the pressure terms in 19) and 12) can be identified while the densities are related by

$$20) \quad \bar{\rho}_a = \rho_a \left( 1 + 3U_a - \frac{U_a}{2} + \frac{v_a^2}{2} \right)$$

or, from 18) and 10),

$$21) \quad \bar{\rho}_a = \rho_a W_a^0 \sqrt{-g} \left( 1 - \frac{U_a}{2} \right)$$

Thus Papapetrou's active gravitational mass is just

$$22) \quad m_a \equiv \int_{R_a} \bar{\rho}_a d^3x = \int_{R_a} \rho_a W_a^0 \sqrt{-g} d^3x - \int_{R_a} \frac{\rho_a W_a^0 \sqrt{-g} U_a}{2} d^3x$$

The similarity of the first term on the right side of 22) to total charge is significant. In fact, if  $P=0$ ,  $\rho W^a$  is conserved so that the argument used to define a constant charge in IV-B above could also be used here. From 17) the second term on

the right can be replaced to this order of approximation by

$$-3 \int_{R_a} \sqrt{-g} P_a W_a^0 d^3x \quad . \text{ Hence}$$

$$23) \quad m_a = \int_{R_a} (P_a - 3 P_a) W_a^0 \sqrt{-g} d^3x$$

or

$$24) \quad m_a = - \int_{R_a} \int_a^\alpha W_a^0 d^3x$$

### E. Measurement of Inertial Mass

#### in Papapetrou's Formalism by Using

#### A Coulomb Force Experiment in Proper Units

The arrangement for measuring an inertial mass associated with  $m_a$  is as follows. First of all, add a spherically symmetric charge distribution to  $m_a$  giving it a total charge  $e$ . Do the same to another mass  $m_b$  much nearer to  $m_a$  than the other bodies in the universe. Further, these particles are assumed to be in an electrically shielded laboratory. Then measure that part of the proper relative acceleration of  $m_a$  to  $m_b$  due to the presence of  $e$ . Determine this acceleration as a function of proper distance between  $m_a$  and  $m_b$  when they are instantaneously at rest.

The limit ( $A_p^i \equiv \dot{x}_p^i$  component of relative proper acceleration)

$$25) \quad \bar{m}_a \equiv \lim_{x_a^i \rightarrow x_b^i} \lim_{e \rightarrow 0} \frac{e(x_a^i - x_b^i)_p}{\frac{\partial \Pi(\lambda_{ab})}{\partial A_p^i}} \frac{1}{\frac{\partial A_p^i}{\partial e}}$$

will then be called the inertial mass of  $m_a$ , provided  $m_b$  is much greater than  $m_a$ <sup>9</sup>

To carry out this program in Papapetrou's formalism, write 7) as

$$26) \quad \int_a^\beta \alpha \beta - \frac{1}{2} \int_a^{\mu\nu} g_{\mu\nu, \alpha} = \sigma_a w_a^\mu \sqrt{-g} F_{\alpha\mu}$$

Equation 26) with  $\alpha = i$  will be integrated over  $R_a$  under the assumption that the rest of the universe is instantaneously at rest. Further, only first order terms in  $m_a$  will be kept. The determination of the electromagnetic field on the right of 26) is based on 2). If all particles are instantaneously at rest,  $g_{0i} = 0$ , to the necessary order and that part of the field due to  $b$  can be written

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9 This is to eliminate the necessity of converting from reduced mass and is only for computational convenience in the example at hand. A more accurate definition taking this reduced mass effect into consideration could easily be made, but its complexity would unnecessarily confuse the point of this example, namely, the independence from the rest of the universe of the relationship between active gravitational mass and a reasonably defined inertial mass.

$$27) \quad \sqrt{-g} F^{i0} = \frac{e(x^i - x_b^i)}{4\pi\lambda_b^3}$$

$$F^{ik} = 0$$

The shielded laboratory walls have eliminated any radiation contributions to 27). In integrating over 26) only that part of the electromagnetic field due to  $b$ , as given in 27), need be kept since the self terms, due to  $a$ , would integrate to zero as a result of the spherical symmetry valid in this approximation. Since only first order results in  $m_a$  are desired, the contribution of  $m_a$  to the metric will be neglected. Further, it will be assumed that  $m_b$ , while much larger than  $m_a$ , is still so small that its contribution to the metric can be neglected in comparison to that of the rest of the universe. Finally, using 18) for the metric, integration of 26) over  $R_a$  gives

$$28) \quad (m_a + \pi_a)(1 + 3U_a) \frac{d^2 x_a^i}{d x_j^2} + m_a G_1^i + m_a G_2^i = \frac{e^2(x_a^i - x_b^i)(1 - 2U_a)}{4\pi\lambda_{ab}^3}$$

where  $\pi_a \equiv \int_{R_a} P_a d^3x$  and  $G_1^i$  and  $G_2^i$  are functions arising from the second term in the left side of 26) and thus are proportional to derivatives of the metric tensor. It is easy to verify that the neglect of the electromagnetic contribution to the metric tensor used in obtaining 28) is justified because of the limit

$e \rightarrow 0$  in the definition.<sup>10</sup> The terms in  $\pi_a$  may not cancel now as they did in Papapetrou's work where  $e=0$  and

$$29) \quad \int_{R_a} \left( P_a - \frac{1}{6} \bar{P}_a u_a \right) d^3x = 0$$

However, here the left side of 29) will be of order  $e^2$ , so that the difference between the  $\pi_a$  terms in 28) and those in the case are of order  $e^2$ , and does not contribute to the terms in the definition.<sup>11</sup>

The conversion of 28) to proper units proceeds precisely as in the transition from II-11) to III-5). The final result is, to the necessary approximation,

$$30) \quad m_a \frac{\partial A_p^i}{\partial e} = \frac{e}{\partial \pi} \frac{(\chi_a^i - \chi_b^i) P}{(\pi_{ab})_P^3}$$

Hence 25) yields

$$31) \quad \bar{m}_a = m_a$$

10 This argument can be sketched as follows. (Changing to proper acceleration and keeping  $e^2$  terms would put 28) in the form

$$(m + \pi + e^2 f) A_p^i g_1 = e^2 g_2^i$$

with  $g_1$  and  $g_2^i$  independent of  $e$ . Thus

$$\frac{\partial A_p^i}{\partial e} = \frac{\partial e g_2^i}{(m + \pi + e^2 f) g_1} - \frac{\partial e^3 f g_2^i}{(m + \pi + e^2 f)^2 g_1}$$

Inserting this in the definition 25) shows that  $f$  could have been set zero.

11 See footnote above.

This then is the required result for the case of a particle representable by a fluid type tensor.

While 31) has been derived only through order one in  $m_a$ , terms of order  $m_a M_{\text{universe}}$  have not been neglected. It may be true that non-zero terms of order  $m_a^2$  will appear on the right side of 31), but these are not really relevant here. The whole purpose of this section and the calculation leading to 31) is to present an example of a relationship between active gravitational mass and a reasonably defined inertial mass that is independent of the rest of the universe. This was done using a unit of charge as a standard so that in these units, using time and charge, the locally measured Newtonian gravitational constant in general relativity is independent of the rest of the universe. This is consistent with Dicke's strong principle of equivalence.<sup>12</sup>

By now the reader is undoubtedly aware that all of the calculations leading to the coordinate free results III-2) or IV-31) could most conveniently have been done directly in a coordinate system in which the background metric has already been "transformed away." A study of this approach to the problem is contained in the next section.

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12 R. H. Dicke, *Science*, 129, 621 (1959). See also R. H. Dicke, *Rev. Mod. Phys.* 29, 355 (1957). *Jour. Wash. Acad. Sci.* 48, 213 (1958). *Am. Jour. Phys.* 28, 344 (1960).

V. Summary and Generalization: Strong and Weak  
Equivalence Principles in General Relativity;  
Other "Mach's Principles"

A. Introduction

Dicke's Strong and Weak Equivalence Principles

This section will be mainly concerned with investigating some of the consequences of the fact that in general relativity the entire gravitational interaction between masses is carried by the metric tensor which can be "transformed away" to any desired degree of accuracy over a sufficiently small neighborhood of any point. This fact leads naturally to the following definition relating a standard physical lab to a mathematical "coordinate patch." A locally almost flat physical coordinate system is one in which test particles of any velocity experience no observable acceleration when there is no matter or radiation within it.

Using this definition, Dicke's<sup>13</sup> strong principle of equivalence can be defined as the assertion that in the absence of non-inertial and non-gravitational forces, the numerical content of experiments performed in a locally almost flat physical coordinate system is independent of any characteristics of the mass distribution in the rest of the universe. It is important to realize that this is a definite extension of such results of the Eötvös experiment as generalized in the weak principle, i. e., the assertion that the acceleration of a test particle in a gravitational field is independent of its mass in the limit as this mass goes to zero. In

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13 R. H. Dicke, Ibid.

other words, the Eötvös experiment suggests that the acceleration effects on a sufficiently small lab of a gravitating body outside it can be at least approximately eliminated by allowing the lab to fall "freely" since it seems to imply that all parts of the lab would fall with very little, if any, relative acceleration. However, it contains nothing to suggest that the only effect of the gravitating body on the lab is accelerative, which is the basis for the strong principle.

A sketch of an argument generalizing the results of Section I-IV above and suggesting the validity of a strong principle in general relativity follows.

B. General Argument Suggesting the Validity of  
A Strong Equivalence Principle in General Relativity

Assume that the stress tensor of matter is the sum of two parts,  $\lambda T_1$  and  $T_0$ ,  $\lambda$  a real number, and  $T_1 = 0$  outside the lab and  $T_0 = 0$  inside it. Further, let the coordinate space-time dimensions of the lab, centered at the origin, be proportional to  $\epsilon$ . Assume that a coordinate system can be chosen in such a way that the differences of the metric tensor from the Minkowskian,  $\eta$ , together with its first two derivatives over the lab are bounded by numbers which go continuously to zero in the limit as both  $\epsilon$ ,  $\lambda$ , go to zero. This is simply a statement of the fact that in the absence of matter within it, the lab is to be locally almost flat. Writing the metric tensor as the sum of two parts  $g(\lambda) + \gamma(\lambda)$  with  $\lim_{\lambda \rightarrow 0} \gamma(\lambda) = 0$  and the variables on which the stress tensor depends as  $\rho$ , satisfying equations of motion,

$$1) \quad f(\rho, g) = 0$$

then

$$2) \quad S^{\alpha\beta}(\rho, g) = T^{\alpha\beta}_\nu(\rho_\nu) \quad ; \quad (S^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{\theta} g^{\alpha\beta} R)$$

and the full field equations reduce to

$$3) \quad S^{\alpha\beta}(\rho, \gamma) = \lambda T^{\alpha\beta}_L(\rho_L) + T^{\alpha\beta}_\nu(\rho_\nu)$$

Hence, within the lab

$$4) \quad S^{\alpha\beta}(\eta, \gamma) = \lambda T^{\alpha\beta}_L(\rho_L) + E^{\alpha\beta} \quad ; \quad f(\rho_L, \eta, \gamma) = H$$

with  $E^{\alpha\beta} \rightarrow 0$  and  $H \rightarrow 0$  as  $\epsilon \rightarrow 0$  and the assumption of sufficient continuity of  $T$  and  $f$  as functions of their arguments has been made. On the other hand, if  $T^{\alpha\beta}_\nu = 0$  the field equations would reduce to 4) with  $E^{\alpha\beta} = 0 = H$ . Hence, assuming continuity of solutions in  $\epsilon$ ,  $\lambda$ , the influence of the rest of the universe on the mass distribution and fields within the lab can be made arbitrarily small by making the lab and masses in it sufficiently small. This is just another way of saying that the "background metric from the rest of the universe can be "transformed away" to any desired degree of accuracy for a sufficiently small lab.

### C. Other "Mach's Principles" in General Relativity

There are, however, other possible statements which might be called "Mach's Principles" and which are valid in general relativity. For example, the universe can influence, over a long enough time, what the inertial mass (as measured in units of charge), of a body is, for example by raising the temperature of a "fluid" or "dust." However, in so doing the active gravitational mass will also be changed and, provided the body can still be contained in a locally almost flat system, the discussion above indicates that the relationship between the two will be expressible in terms involving only the final state of the fluid or dust and excluding any reference to the rest of the universe. Further, it should be noted that this is not a local effect, in the general four dimensional sense used in the strong principle; since for any given state of the universe there is a lower bound to the time required to produce an observable effect.

Another possible Mach's principle might be suggested by the statement that inertial and gravitational forces have a common formal origin in general relativity. For example, for a test particle of mass  $m$  and velocity  $W^\alpha$ ,

$$5) \quad F^\mu = -m \Gamma_{\alpha\beta}^\mu W^\alpha W^\beta$$

might be identified with the gravitational force. On the other hand, this term transforms just as an inertial force should, i. e., in going to a relatively accelerated system, the acceleration enters

$F^\mu$  linearly. Thus  $F^\mu$  might also be identified with "inertial force."

Inertial coordinate systems would then be those in which  $F^{\mu}$  vanishes or equivalently, those in which "free," uncharged test particles are unaccelerated. This coincides with the definition of locally almost flat coordinate systems above. Another way of saying this is that the locally almost flat or inertial coordinate systems are those in which the total gravitational force vanishes. Thus, since Einstein equations, together with suitable boundary conditions, relate the metric tensor to the mass distribution, general relativity does predict the state of motion (up to a velocity translation) of the inertial frames relative to the rest of the universe. However, once it is required that fundamental, standard experiments be done in such frames, the rest of the universe cannot, in general relativity, influence their results.

## PART II

VI. IntroductionA. Strong Versus Weak Equivalence Principles;  
Gravitational "Constant"

That general relativity satisfies the strong as well as the weak principle of equivalence has been indicated by the discussion in part one. There it was shown that in general relativity it is not only true that nearby masses give all test particles instantaneously at rest in a small shielded lab approximately the same acceleration (weak principle) but also that this acceleration is the only effect of these masses detectable within such a lab, provided it is sufficiently small (strong principle).

The purpose of part two is to consider a theory that explicitly violates the strong principle of equivalence but is otherwise very similar to Einstein theory. In particular, the weak principle will be rigorously satisfied.

While the Eötvös experiment done with sufficient accuracy might be taken as an indication of the validity of this weak principle, it says nothing at all about the strong principle. In fact, this strong principle, on which general relativity is based, has not been experimentally verified. On the contrary, there are reasons<sup>14</sup> for suspecting that it is not true. Hence a theory violating it might assist much needed experimental investigations.

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14 See e.g. Part I pages 1-2, and the discussion following in the text. These arguments suggest a violation of the strong principle mainly through a varying gravitational "constant." Other possibilities include varying dielectric and/or fine structure "constants." See e.g. P. Jordan, Schwerkraft und Weltall, Braunschweig (1955), Second edition; R. H. Dicke, Science 129 621 (1959), A. J. Phys. 28 344 (1960).

In discussing the strong principle and local standard experiments testing it, it is important to recall that all experimental results are basically ratios of numbers. Some of these ratios contain standard "units" in an essential way. For example, the statement, "the velocity of light,  $C$ , equals one," makes no physical assertion beyond stating that the units of time and length have been chosen so that the ratio  $\frac{\Delta l}{\Delta t}$  for a light ray interval  $(\Delta l, \Delta t)$ , in these units is one. Nature provides other basic quantities, e.g., Planck's constant  $\hbar$ , the mass  $m_e$  and charge  $e$  of the electron, the visible mass in the universe,  $M_V$ , the age of the universe  $T$ , and the Hubble radius  $R_H$ , whose numerical values are again obviously dependent on units. However, they do combine into numbers (Eddington numbers) such as

$$\frac{\hbar c}{e^2} \quad ; \quad \frac{R_H}{c T} \quad ; \quad \frac{K M_V}{c^2 R_H}$$

$$1) \quad \frac{R_H}{(\hbar / m_e c)} \quad ; \quad \frac{T}{(\hbar / m_e c^2)} \quad ; \quad \frac{e^2}{K m_e^2}$$

$$\frac{M_V}{m_e}$$

that are dimensionless and have values which are independent of the units used in measuring the individual factors. Hence, statements to the effect that any or all Eddington numbers are dependent (or independent) of when and where they are measured are definite, well defined statements subject to experimental

testing. For example, at least one of these numbers,  $\frac{\Gamma}{(\hbar/m_e c^2)}$  might certainly be expected to depend on when it is measured.

At first glance 1) appears to be a rather motley, if "fundamental" collection of numbers. Actually their presently observed terrestrial values seem to suggest a relationship between them,<sup>15</sup> In fact, the numbers in the nth line of 1) are very roughly equal to  $10^{40(n-1)}$ .

This is a very big span of numbers. Accepting these values as purely coincidental might mean overlooking important physical relationships. Dirac<sup>16</sup> conjectured that a comprehensive theory might connect these numbers. This connection might be approximately a simple algebraic one. The numbers in the first, second and third row are approximately equal to the zeroth<sup>17</sup>, first and second power respectively of the same number. The most obvious choice for this independent variable might be  $\frac{\Gamma}{(\hbar/m_e c^2)}$ , the age of the universe in "atomic" units.

15 P. A. M. Dirac, Proc. Roy. Soc. (London) A165 199 (1938).  
R. H. Dicke, Rev. Mod. Phys. 29 355 (1957).

16 P. A. M. Dirac, Ibid.

17 In these order of magnitude arguments there is little difference between zeroth power and logarithm. For  $\alpha$  the logarithmic dependence has been suggested by Landau and investigated by Dicke, Science 129 621 (1959) in the context of Dirac's cosmology. Thus

$$\alpha \approx \ln \left[ \frac{\Gamma}{(\hbar/m_e c^2)} \right] \approx \ln \left( \frac{\hbar c}{k m_e^2} \right)$$

The resulting time variation in the Eddington numbers could be accounted for in several ways. Quantities such as  $\hbar$ ,  $m_e$ ,  $c$  could be allowed to vary. This would entail, of course, a revision of quantum theory, elementary particle studies and electromagnetic theory. On the other hand, these quantities could be fixed, by definition, and thus provide a natural set of units. Setting  $\hbar = m_e = c = 1$ , the values of  $M_V$ ,  $R_H$ , and  $K$  would then vary with time,  $t$ ,

$$\frac{K M_V}{R_H} \approx t^0 = \text{constant}$$

$$R_H \approx t^1$$

$$\frac{1}{K} \approx t^1$$

$$M_V \approx t^2$$

The first line of 2) can be rewritten as

$$3) \quad \frac{1}{K} \approx \frac{M_V}{R_H}$$

The right side of 3) might be interpreted as some sort of sum of  $m/r$  over "visible," i.e., causally connected, masses and radii. A relation similar to this was obtained from dimensional arguments in part one, I-B, pages 1-3.

In part two a theory containing a varying "constant" will be considered but not precisely in the "units" mentioned above. Nothing will be said about the electron mass,  $m_e$ . The velocity of light will still be defined to be unity but now, as in

part one, basic units of time (e.g. an atomic period) and charge (e.g. the electron's) will be assumed available, and the latter made to correspond with the conserved total charge mathematically obtained from the current density. A standard Coulomb-law type experiment then provides a method for measuring of inertial mass.<sup>18</sup> It is in these units that calculations will be made of the locally measured gravitational "constant" as a function of the mass distribution in the universe. Whether or not  $h$  and  $m_e$  expressed in these units are really constant is beyond the scope of this paper.

Given the desirability of investigating a theory containing a gravitational constant dependent on the mass distribution of the universe, how might such a theory be constructed? In particular, how might  $K$  be determined?

It is clear that  $K$  must be a function of some invariant if it is to be dependent only on where and when it is measured. The only field quantities available in general relativity are the metric and electromagnetic fields. Since it would be required that all matter, even uncharged, should contribute, it is clear that in any theory similar to standard general relativity the main contribution would have to come from the metric as opposed to the electromagnetic field. The only non-trivial invariants formed from the metric would depend on at least its second derivatives. These, however, would be too sensitive to local matter in any Einstein type theory. Further, these invariants contain only dimensions of length, so that the function expressing  $K$  in terms of them would have to be dimensional itself.

Hence it seems likely that a scalar field of which  $K$  would be a function must be added to general relativity.

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<sup>18</sup> See Part One IV-E, pages 21f. and Part Two IX-C, pages 71 f.

### B. The Variational Principle and Field Equations

Denoting the scalar field by  $\phi$ , a hint as to what function  $K$  will be of  $\phi$  might be taken from Part One I-B pages 1 f. and Part Two VI-A pages 31 f. above. Specifically some relation such as

$$4) \quad \frac{1}{K} \approx \sum \frac{m}{\lambda}$$

might be expected. Hence a likely candidate for a generally covariant field equation might be

$$5) \quad \square \frac{1}{K} = -\rho \quad (\square f \equiv f^{;\mu}{}_{;\mu})$$

with  $\rho$  proportional to some invariant "mass" density. Hence,  $\frac{1}{K}$  appears as a reasonable choice for the field variable  $\phi$ .

This is borne out in an analysis of a variational principle (VII pages 48 f.). The basic assumptions used in obtaining this variational principle are that the equations of the matter variables are to be formally the same as in general relativity and that the Lagrangian for the metric is to be proportional to the curvature scalar. Thus the weak principle of equivalence is still rigorously satisfied and the field equations are linear in the second derivatives of the metric tensor. Further, the entire theory of matter and electromagnetism is left unchanged and the Coulomb determination of inertial mass used in general relativity (Part One IV-E pages 21 f.) is the same.

These conditions narrow the choice of variational principle considerably, although there still remains some freedom

in the choice of Lagrangian for  $\phi$ . This is eliminated by choosing the simplest Lagrangian consistent with a dimensionless coupling constant and a field equation for  $\phi$  of the form given by 5).

The field equations obtained from the finally chosen variational principle are very similar to the standard Einstein equations, but with a modified source tensor, (VII-B, pages 49 f.). This source tensor is the product of  $\frac{1}{\phi}$  with the sum of the ordinary mass tensor and a tensor obtained from  $\phi$  and its derivatives. Thus from the field equations  $\frac{1}{\phi}$  does appear to play the role of the gravitational "constant." However, this is not sufficient to prove that  $\frac{1}{\phi}$  will be the locally measured Newtonian gravitational "constant." (See VI-E, pages 41 f.).

The extra function,  $\phi$ , introduces more freedom in the initial value or boundary value problem than is available in general relativity. There is some difficulty in eliminating this freedom so as to obtain from the field equations a value of  $\phi$  something like  $\sum \frac{m}{r}$ . (XIII, pages 87 f.).

It is seen that the Einstein equations are approached in the limit of large coupling constant  $\omega$  (VII-D, pages 53 f.).

The weak field approximation is carried out to demonstrate the necessary weak field Einstein and Newtonian limit and to get some information about the relation between  $\phi$  and the locally measured gravitational "constant." (VII-E, pages 55 f.).

### C. Jordan's Work

Field equations very similar to those discussed above have been suggested by Jordan<sup>19</sup> and investigated by him and

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19 P. Jordan, Ibid.

others. Starting from equations obtained from his five dimensional projective "unified field theory" he studies general types of Einstein-like field equations containing a function,  $\chi$  which he seems to interpret as the gravitational constant (VIII-B pages 60f.). However, he is not too thorough or clear in his analysis of how matter contributes to the field equations, or what quantity associated with his "matter tensor" is to be interpreted as inertial mass. It is to be emphasized that the introduction of inertial mass is necessary before any statements can be made about what the locally measured gravitational "constant" is. Fierz<sup>20</sup> has also pointed out the ambiguities involved in Jordan's treatment of mass (VIII-E pages 64f.)

Others, notably Just<sup>21</sup>, have explicitly dealt with matter as a source in the field equations, and evaluated the constants in vacuum solutions in terms of integrals over "matter" variables. How these are related to inertial mass is not clear, however.

The main differences between this paper and the work of Jordan and others can be summarized as follows (VIII-G pages 66f.). First, the field function is here taken to be the reciprocal of the gravitational constant rather than the constant itself. Secondly, in this paper matter is explicitly kept in the theory in a way consistent with the weak equivalence principle. Finally, a locally measured Newtonian gravitational constant is defined and an attempt made to relate this to  $\phi$  and through  $\phi$  and the field equation to the structure of the universe.

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20 M. Fierz, *Helv. Phys. Acta* 29, 128 (1956).

21 K. Just, *Z. Phys.*, 140 524 (1955).

#### D. Infeld's Equations of Motion Through Second Order

In order to get any indication at all of how matter contributes to the locally measured gravitational constant, it is necessary to carry the equations of motion to second order at least. This was pointed out in Part One pages 8 to 10. The method of Infeld<sup>22</sup> has been chosen for this purpose. For the purposes of this paper Infeld's matter tensor, which is a type of delta function, is taken to represent bodies whose sizes are small compared to their separation.<sup>23</sup> In integrating products of functions, such as the components of the metric tensor with components of the matter tensor, the latter is supposed to act like a delta function except that it eliminates self contributions, i. e., functions singular at the origin of the delta function (IX-B pages 69 f.)

Actually, such integrals are used only in two places. One is in the field equations where the components of the matter tensors, multiplied by components of the metric, appear as sources. Hence, self terms can be assumed to be already contained<sup>24</sup> in the numbers such as  $\mu \xi^{\alpha} \xi^{\beta}$  which are defined to be equal to these integrals. Of course, it is further assumed that the  $\xi^{\alpha}$  appearing here correspond to the observed velocities of the bodies, so that there is really only one free parameter,  $\mu$ , in which to "absorb" self effects. This

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22 L. Infeld, Rev. Mod. Phys., 29 398 (1957).

23 See Part One page 18 where a similar restriction is used in Papapetrou's method.

24 See the relation between Papapetrou's mass "density" and the standard choice for fluid density, Part One IV-D equations 20) and 21), page 20.

does not seem to be a strong assumption but merely indicates that in the large the body must act like a nonspinning point particle. The second place where these integrals are encountered is in the equations of motion. The terms dropped are self forces, and their neglect corresponds to the assumption that they are balanced by internal pressures or at any rate do not contribute to the observed gross motion of the body.

Of course the number,  $\mathcal{M}$ , introduced in terms of integrals of the matter tensor must be related to an observed mass. This is done by placing a small charge on the body and calculating its equation of motion in an external field. Again self effects due to the charge are neglected. The result is that  $\mathcal{M}$  plays the role of inertial mass (IX-C pages 71 f.). This is directly comparable to the procedure employed in the case of Papapetrou's method (IV-E pages 21 f.).

Finally, the geodesic equations of motion of a test particle through second order are obtained. In the Einstein limit ( $\frac{1}{\omega} = 0$ ) these equations are identical to those obtained by Papapetrou. Further, Infeld's metric and Papapetrou's differ only in fourth order. To convert the coordinate description of the motion of the particles to proper units corresponding to physical measurement it is only necessary to use the metric up through third order. This is so because the distances, velocities and accelerations to be converted are contained only in combinations already of order 2 and the expressions for the accelerations are correct only through order 4. Thus, in the Einstein limit at least, the two methods give identical observable results.

This might also be expected even if  $\frac{1}{\omega} \neq 0$ . Hence it seems very likely that the fluid type matter tensor of Papapetrou-Fock would give the same observable results as the singularities of Infeld in the case of non-constant  $\phi$ .

E. Definition of Locally Measured  
Newtonian Gravitational Constant  
And Its Evaluation Through Second Order

To apply these results to the calculation of a locally observed Newtonian gravitational "constant," it is necessary to define this "constant" by describing the method of measuring it. As noted above, it is not sufficient to say that since in the field equations  $\frac{1}{\phi}$  seems to play a role similar to  $K$  in the Einstein case, it will correspond (as  $K$  does in the Einstein case, III-B page 11 ) to the result of a proper local Cavendish experiment.

First of all, it is assumed that a basic unit of time (or length) is available which corresponds to the mathematical proper time (or length) given by the metric and that the unit of length (or time) is obtained by the requirement that the velocity of light be one. The path of a light ray is assumed to be a null geodesic. Further, the physical measurement of inertial mass is assumed to be consistent with the mathematical one chosen either in the Infeld (IX page 68 f) or the Papapetrou-Fock (IV pages 13 f) method and physically defining the "masses" obtained from their matter tensor.

The actual definition of  $K_E$ , the effective gravitational constant, is then based on a comparison of the actual motion

of a test particle, to that predicted on Newtonian basis (X-B page: 75). To obtain this Newtonian limit it is necessary to assume that the test particle and gravitating mass are small. Further to make the experiment local and to eliminate curvature effects they must be brought close together.

The result, based on the equations of motion through second order, is that the universe does contribute to  $K_E$  (X-B page: 75). In fact, to first order in  $\frac{K_0 m}{\lambda}$ ,

$$6) \quad \frac{1}{K_E} \approx \frac{1}{K_0} - \frac{1}{8\pi(\omega-2)} \sum \frac{m}{\lambda}$$

Einstein's arguments might well be used here. Namely, if some of the universe ( $\sum \frac{K_0 m}{\lambda} \ll 1$ ) contributes to  $\frac{1}{K_E}$ , then all of it probably does. In particular, to make the first term,  $\frac{1}{K_0}$ , in 6) the result of a sum similar to the second for the whole universe (where  $\sum \frac{K_0 m}{\lambda} \approx 1$ ),  $(\omega-2)$  must be negative.

It should also be noted that the "background value,"  $K_0$ , appearing in 6) is not the asymptotic value of  $\phi$  but  $\frac{\partial\omega-4}{\partial\omega-3}$  times this value. This follows from the fact that in the calculations of any local experiment  $\phi$  enters in more than one way. Not only does  $\frac{1}{\phi}$  multiply the matter tensor but  $\phi$  also contributes additive terms to the "effective" matter tensor in the analogue of the Einstein equations. The additive terms, containing first and second derivatives of  $\phi$ , arise from local matter itself. Hence, any gravitating mass contributes to the metric tensor twice, once directly through the matter tensor and then through the  $\phi$  terms.

### F. The Heckmann Solution And Three Standard Tests

After obtaining some information from approximate solutions, the next step is to look for exact solutions. The most obvious place to look for these might be in the static spherically symmetric vacuum case. Jordan<sup>25</sup> has stated such a solution, for his field equations in Schwarzschild like coordinates, due to Heckman. In the vacuum, these field equations are formally identical with those considered in this paper, requiring only the replacement of  $\chi, \dot{\chi}$  by  $\phi, \dot{\phi}$ . Hence this solution is valid here and use is made of it to obtain the outcome of the three standard tests (XI-B page 78).

However, care must be taken in evaluating the constants in the vacuum solutions in terms of the inertial mass of the "singularity." Jordan's procedure here is not clear and in any case, since his interpretation of mass and  $\chi(r, \phi)$  is different from that used in this paper, a revaluation is necessary (XI-B page 78).

### G. Exact Spherically Symmetric

#### Static Vacuum Solution in Isotropic Coordinates

The Heckmann solution, while useful, is difficult to interpret and analyze since to express  $\phi$  and the metric components in terms of elementary functions a parametric representation must be used. Since the use of isotropic coordinates often simplifies situations in general relativity, Misner<sup>26</sup> suggested it might do so here also. This is in fact true. This

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25 P. Jordan, Ibid., p. 172

26 C. W. Misner, Private Communication.

solution is given in XII, pages 79 f.). The four "branches," corresponding to different ranges of the independent constants, are restated together for convenience in XII-F, pages 85 f.

#### H. Boundary Conditions

The general isotropic solution mentioned above contains essentially three independent constants (in addition to two trivial ones associated with space and time units). Matching to an interior solution (which has yet to be obtained in exact form) would eliminate two of them. The third, which simply multiplies  $\phi$ , would still be undetermined, however.

This indeterminacy is an essential obstacle to the program of obtaining the gravitational "constant" from the structure of the universe.

To overcome this difficulty Dicke<sup>27</sup> has suggested that boundary conditions be imposed on  $\phi$ . For no matter at infinity he suggests  $\phi \rightarrow 0$ . This is motivated by  $\phi \sim \Sigma \frac{m}{r}$  so that in the absence of matter,  $\phi \rightarrow 0$ . Further, it has been conjectured that a Mach's principle might indicate that in the absence of matter the field equations should become meaningless. Equation VII-8) page 51, shows that  $\phi = 0$  would render the field equations indeterminate.

The simplest type of universe in which to study this problem is that containing only a static, spherically symmetric mass shell. The analogous potential problem for this case in electrostatics will be considered (XIII-B, page 87). It is found that the potential for this problem is not fully determined

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27 R. H. Dicke, Private Communications.

in terms of the shell parameters. There is the well known additive constant available. In electrostatics this can be eliminated by requiring that the potential be zero at infinity. If this is done, the potential inside will be a constant proportional to the total charge of the shell divided by its radius. This is precisely the result desired for the scalar field related to the gravitational constant. (See I-B, pages 1 f., VI-B pages 36 f.).

Since in the weak field case, to lowest order,  $\phi$  does satisfy a flat space Poisson equation it might be thought that these electrostatic results could be directly carried over. However, this is found not to be true. Essentially, the difficulty is that  $\phi \rightarrow 0$  at infinity is inconsistent with the weak field assumptions since it requires  $\frac{1}{k} \sim \frac{M}{R}$  in violation of  $\frac{KM}{R} \ll 1$  (XIII-C pages 89 f.).

Hence exact solutions should be used. The specific problem studied in section XIII pages 87 f. is that in which the matter tensor is static spherically symmetric, diagonal and vanishes outside a range  $R_1 < \lambda < R_2$  apart from relatively small masses near  $\lambda = 0$ . The solutions inside,  $\lambda < R_1$ , and outside,  $\lambda > R_2$ , will be of the form given in XII-F pages 85 f. Hence there will be six independent constants, three inside and three outside. The masses near the origin and the shell itself would be expected to determine five of these six constants. To eliminate the remaining one consider the boundary condition  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$ . A glance at XII-F pages 85 f. shows that this can only be satisfied for a choice of constants in the solution outside appropriate to type III. This is a particular solution so that the extra constant has been eliminated. Unfortunately, however, type III

requires positive  $\omega$ , contradicting the positive contribution of local matter. (VI-E pages 41 f. ; X-B pages 75 f.).

Actually, an analysis of this case shows that  $\phi$  can go to zero as  $r$  goes to infinity only if the trace of the stress tensor is negative (corresponding to  $p > \frac{1}{3}\rho$  for a fluid) (XIII pages 87 f. ). Nevertheless, an approximation to the behavior of an interior solution will be made (XIII-H pages 96 f.) and the resultant determination of the local  $K_E$  near the center in terms of the radius, mass and total pressure of a spherical ball is estimated for this case,  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$ .

The only other possibility is  $\phi \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$  with  $\lambda_0$  a finite radius. At best this would result in a determination of the local  $K_E$  near the center in terms of the shell parameters and  $\lambda_0$ , this latter being still arbitrary. Even at this it is found that such a boundary condition would require pressure terms bigger than densities (XIII-F, pages 93 f.).

Of course in a more realistic universe the boundary condition would be  $\phi \rightarrow$  a cosmological solution. However, there is still some indeterminacy in the latter, e.g. the initial value at some time, Thus some global boundary condition (e.g. that space be closed) might still be needed. Although this problem will not be discussed in this paper, some information will be obtained about the cosmological solution in Section XIV pages 100 f.

### I. Cosmological Solution

Section XIV pages 100 f. will contain a very brief discussion of the Friedman universe in general relativity and an evaluation of the Eddington numbers (VI-A pages 31 f. ) in terms of them.

The analogous equations for the  $\phi$  field case are stated and briefly investigated. Unfortunately, the exact general solutions are not available, although some important special solutions can be exhibited together with the associated Eddington numbers.

The fact that  $\omega$  must be large (XI-B pages 78 f.) suggests that some information about the general solution might be obtained by expanding it in a series in  $\frac{1}{\omega}$ . The first few terms and the associated Eddington numbers fail to simultaneously satisfy all of Dirac's conjectures

$$7) \quad \begin{aligned} \frac{1}{k} &\sim \phi \sim t \\ R_H &\sim t \\ M_V &\sim t^2 \end{aligned}$$

Thus the present state of the cosmological problem is highly unsatisfactory and will require much more study.

The lowest order effects of a slowly varying gravitational "constant" on planetary orbits is obtained in XIV-O, pages 122 f.

### J. Conservation Laws

The field equations VII-B pages 49 f. being essentially Einstein equations with modified matter tensor, a first attempt to obtain a conservation law might be based on the procedure used in the Einstein case. This yields a conserved quantity, giving a total "mass" expressed in length units and corresponding to some sort of average gravitational "constant" times mass. The result, for the static spherically symmetric case is a number proportional to the "gravitating radius" of the singularity (XV-B pages 124 f.).

Another method would be to apply the canonical procedure directly to the variational principle. This proves too unwieldy, however. (XV-C, pages 127f.).

Finally, a conservation law giving a total "mass" in units of mass can be obtained by writing the conserved affine tensor as the divergence of an antisymmetric affine tensor. The result is a number proportional to the "gravitating radius" times the asymptotic value of  $\phi$ . (XV-D, pages 128f.).

## VII. Variational Principle and Field Equations

### A. Introduction—Criteria

The possibility of a varying gravitational "constant" has been discussed by Dirac<sup>28</sup>, Jordan<sup>29</sup>, and particularly with respect to Mach's principle by Dicke<sup>30</sup>. The idea is to weaken the strong principle of equivalence through the effective gravitational constant. As noted in Part One III-B, page 11, the standard interpretation of general relativity relates the gravitational constant entering the field equation with the locally measured Newtonian constant in an unchangeable manner. The most straightforward approach to an alteration of this result is thus to relate the gravitational "constant" entering the field equations to a field quantity determined by the mass distribution in the universe. The object of Part Two is to describe and partially analyze such a formalism.

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28 P. A. M. Dirac, Ibid.

29 P. Jordan, Ibid.

30 R. H. Dicke, Ibid.

In choosing a variational principle violating the strong principle of equivalence by the introduction of a varying gravitational "constant," it seems desirable to satisfy at least two conditions. First, the variational principle must be similar to the standard Einstein principle. In other words, since the Einstein equations do agree with the observed data fairly well, any extension of the theory might be expected to be formally similar. Second, the variational principle must be consistent with the weak principle of equivalence which is just a generalization of the results of the Eötvös experiment.

To satisfy this second condition it will be required that the operational definition of inertial mass be prescribed in a manner formally independent of the structure of the universe. The stress tensor of ponderable matter will be identified formally and interpretatively with that of general relativity. For example, the equation of a test particle must be a geodesic and a procedure for obtaining an inertial mass for given matter Lagrangian or stress tensor such as in IV can still be used. Clearly, however, the gravitational determination of mass feasible in general relativity and discussed in Part One must be excluded here.

#### B. Variational Principle and Field Equations

Expressing the standard variational principle as

$$1) \quad \delta \int \mathcal{D}^4x \sqrt{-g} (R + K L_m) = 0$$

with  $L_m$  the matter Lagrangian, it is obvious that the simple replacement of  $K$  by a variable would violate the above requirements, since the matter equations would be

formally altered. However, division by  $K$  yields an equation amenable to such a substitution. For this purpose,  $\frac{1}{K} \rightarrow F(\phi)$ , with  $F$  some functional of a field,  $\phi$ , taken here to be a scalar, appears as the obvious and simplest choice so that the total variational principle will be taken as

$$2) \quad \delta \int d^4x \sqrt{-g} (F(\phi)R + L_m + L_\phi) = 0$$

with  $L_\phi$  some Lagrangian for  $\phi$ . The requirement that the field equation for  $\phi$  be second order gives

$$3) \quad L_\phi = L_\phi(\phi, \phi_{,\mu})$$

Apart from this, there seem to be few restrictions on  $L_\phi$ . If  $F(\phi) = \phi$ , the standard choice

$$4) \quad L_\phi = \omega \phi_{,\mu} \phi_{,\nu} g^{\mu\nu}$$

while giving the wave equation for  $\phi$  with  $R$  as source, requires a dimensional coupling constant,  $\omega$ . On the other hand, if  $L_\phi$  is taken homogeneous of degree two in  $\chi^\mu$  and degree one in  $\phi$ , the coupling constant is dimensionless and if in particular,

$$5) \quad L_\phi = \omega \frac{\phi_{,\mu} \phi_{,\nu} g^{\mu\nu}}{\phi}$$

the field equations for  $\phi$  reduce to the wave equation with the trace of the matter tensor for source. The variational

principle will be thus taken to be

$$6) \quad \delta \int d^4x \sqrt{-g} \left( \phi R + L_m + \omega \frac{\phi_{, \mu} \phi_{, \nu} g^{\mu \nu}}{\phi} \right) = 0$$

Here  $\phi$  has the dimensions of reciprocal gravitational constant,  $\omega$  is a dimensionless constant number. The field equations associated with this principle become

$$7) \quad \delta_m \int d^4x \sqrt{-g} L_m = 0$$

$$8) \quad \phi \Sigma_{\alpha\beta} = T_{m\alpha\beta} + \phi_{, \alpha} \phi_{, \beta} - g_{\alpha\beta} \square \phi - \frac{\omega}{\phi} \left( \phi_{, \nu} \phi_{, \alpha} - \frac{1}{2} g_{\alpha\beta} \phi_{, \nu} \phi_{, \lambda} \right)$$

$$9) \quad \omega \left( \frac{\partial \square \phi}{\phi} - \frac{\phi_{, \lambda} \phi_{, \lambda}}{\phi^2} \right) = R$$

in which  $\delta_m$  signifies variation with respect to pertinent matter variable,  $T_{m\alpha\beta}$  is the usual matter tensor and

$$10) \quad \Sigma_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \quad ; \quad \square \phi \equiv \phi_{, \lambda}{}^{, \lambda}$$

It is immediately clear that 8) and 9) are equivalent to 8) and

$$11) \quad (2\omega - 3) \square \phi = -T_m \quad ; \quad T_m \equiv T_m{}^\alpha{}_\alpha$$

The verification of the conservation equation

$$12) \quad T_{m\alpha}{}^\beta{}_{; \beta} = 0$$

is obtained from 8) and 9) by straightforward calculation

using the Bianchi identities

$$13) \quad \sum_{\alpha} \overset{\beta}{j}_{\beta} = 0$$

and

$$14) \quad \phi_{j^{\alpha}} - \phi_{j^{\alpha}} - \phi_{j^{\alpha}} = -R_{\beta\alpha} \phi_{j^{\alpha}}$$

### C. Initial and Boundary Value Data;

#### Exceptional Case, $\phi=0$

The study of the existence of solutions and the initial value problem for these equations is analogous to that for the Einstein set. In fact, if  $\phi \neq 0$ , dividing 8) by  $\phi$  yields the Einstein equations with modified matter tensor. Further, 11) is independent of the second derivatives of the metric tensor. Hence, appropriate initial value data would consist in the values of  $g_{\alpha\beta}$ ,  $\phi$ , and their first normal derivatives on a three-surface (with non-null normal), together with the necessary quantities for  $T_{\alpha\beta}$ , all restricted to satisfy the " $(\alpha)$ " set of 8).

Thus considerably more data at the initial time is needed to specify the future course of a system than in general relativity. In fact, two functions must be measured everywhere, the value of the gravitational "constant," which could only be a constant (but arbitrary) number in general relativity and its first time derivative. The physical significance of this last function is rather hard to grasp and

there is no analogue for it in general relativity. This extra freedom must somehow be eliminated in any attempt to determine the gravitational constant from the mass distribution of the universe in a way consistent with Dirac's conjecture (see Section VI-A and B pages 31 f. ), The obvious candidate would be a boundary condition  $\phi \rightarrow 0$  outside all matter. This will be considered in Section XIII, pages 87 f.

On the other hand, if  $\phi$  is zero the left side of 8) becomes zero so that there is no determination of the second derivatives of the metric tensor. In other words, the field equations break down in this case. Interpreting  $\phi = \text{zero}$  over a region, from  $\phi \sim \Sigma \frac{m}{r^2}$ , as an indication of no net influence of matter in this region, this result might suggest that in the absence of matter the field equations are indeterminate. This has been mentioned as a possible statement of a Mach's Principle.

#### D. Einstein Equation for Large $\omega$

At first glance, it appears that the Einstein equations are approximated in some sense for large  $\omega$ , since 11) is consistent with  $\phi \xrightarrow{\omega \rightarrow \infty} \text{constant}$  and this last limit inserted in 8) gives the Einstein equations. In fact, setting  $\phi = \text{constant} + \frac{\psi}{\omega}$ , the equations are consistent with  $\lim_{\omega \rightarrow \infty} \frac{\partial \psi}{\partial \omega} = 0$  or  $\lim_{\omega \rightarrow \infty} \frac{\psi}{\omega} = 0$ , while the difference between 8) and the Einstein equations are of order  $\frac{\psi}{\omega}$  in this case. However, it would not be correct to say that these equations become approximately fully equivalent to those of Einstein for large  $\omega$ , since the addition of the function  $\phi$  has

effectively rendered the equations less stringent in their determination of the metric and matter tensors. As a counter example, it is easily observed that the set

$$g_{\alpha\beta} = \eta_{\alpha\beta} \quad ; \quad \phi = \frac{\rho(\omega)^2}{2\omega}$$

$$15) \quad \pi_{00} = \rho \quad ; \quad \pi_{0i} = 0 \quad ; \quad \pi_{ij} = \rho \delta_{ij} \left( \frac{\omega-1}{\omega} \right)$$

$\rho = \text{constant}$

is a solution to the equations 8) and 9). On the other hand, for fixed  $\rho$  and  $\omega$ , it is clear that 15) cannot be arbitrarily closely approximated over a neighborhood by a solution to the Einstein equations. However, the above discussion might be taken as an indication of the converse, namely, that any solution of the Einstein equations, over a compact neighborhood, can be arbitrarily closely approximated by a solution to the extended set for sufficiently large  $\omega$ .

Later comparison (XI, pages 77f.) of perihelion rotations with Einstein results will require

$$16) \quad |\omega| \gtrsim 10$$

Further, using  $\phi \sim \sum \frac{m}{r}$  it would seem likely that the addition of neighboring masses might increase  $\phi$ . However, in VII-E, pages 55f. below, it will be seen that this requires

$$17) \quad (2\omega - 3) < 0$$

Thus  $\omega \lesssim -10$  seems to be the reasonable range.

### E. Weak Field Approximations

To obtain the approximate linearized equations, set

$$18) \quad g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \\ \phi = \frac{1}{K}(1 + \Psi) \quad ; K = \text{constant of dimensions } LM^{-1}$$

Retaining only terms linear in  $h$  and  $\Psi$  the field equations reduce to

$$7a) \quad \nabla_m \Gamma^{\alpha\beta}{}_{,\beta} = 0$$

$$8a) \quad \frac{1}{2K} (-\eta^{\mu\nu} \gamma_{\alpha\beta,\mu\nu} + \sigma_{\alpha\beta} + \sigma_{\beta\alpha} - \eta_{\alpha\beta} \eta^{\mu\nu} \gamma_{\mu\nu}) = \nabla_m \Gamma^{\alpha\beta} + \frac{1}{K} (\Psi_{,\alpha\beta} - \eta_{\alpha\beta} \eta^{\mu\nu} \Psi_{,\mu\nu})$$

$$11a) \quad (2\omega - 3) \Psi_{,\mu\nu} \eta^{\mu\nu} = -K \nabla_m \Gamma^m$$

Where

$$19) \quad \gamma_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} \eta^{\mu\nu} h_{\mu\nu} \quad ; \quad \sigma_{\alpha} \equiv \gamma_{\alpha\beta,\rho} \eta^{\beta\rho}$$

The coordinate conditions  $\sigma_{\alpha} = \Psi_{,\alpha}$  are compatible with 7a), 8a), 11a) since the "small" transformation  $x^{\alpha} \rightarrow x^{\alpha} + v^{\alpha}$  with  $v_{\alpha,\mu\nu} \eta^{\mu\nu} = \sigma_{\alpha} - \Psi_{,\alpha}$  leads to a system in which they are satisfied. In fact, the solution can be specified as

$$20) \quad \gamma_{\alpha\beta} = \delta_{\alpha\beta} + \eta_{\alpha\beta} \Psi$$

with

$$21) \quad \frac{1}{2K} \eta^{\mu\nu} \gamma_{\alpha\beta, \mu, \nu} = -\frac{\Gamma}{m} \alpha\beta ; \quad \gamma_{\alpha\beta, \rho} \eta^{\rho\sigma} = 0$$

and  $\Psi$  satisfying 11a). Hence the metric tensor is the sum of two parts,  $g_{\alpha\beta} = g_{\alpha\beta}^0 - \eta_{\alpha\beta} \Psi$ , with  $g_{\alpha\beta}^0$  satisfying the usual Einstein equations and coordinate conditions. Thus, the radial acceleration of a test particle at distance  $r$  from a single small mass  $m$  is

$$22) \quad \frac{1}{2} g_{00, r} = -\frac{K_0 m}{8\pi r^2} \left(1 - \frac{1}{2\omega-3}\right) = -\frac{K_0 m}{8\pi r^2} ; \quad K_0 \equiv \left(\frac{2\omega-4}{2\omega-3}\right) K$$

These equations, apart from demonstrating the required Newtonian limit, provide an interpretation of  $K, \omega$  in terms of the observed effective gravitational constant,  $K_0$ , and also illustrate the manner in which the Einstein limit can be approached for large  $\omega$ . In this approximation, qualitative differences with the Einstein results are contained only in velocity effects. Thus, for example, for a small gravitating mass density  $\rho$  with velocity  $\frac{d\alpha^i}{d\alpha^0} = v^i \ll 1$ , a slow moving test particle at  $\alpha^i$  follows the geodesic approximately given by

$$23) \quad \begin{aligned} \ddot{\alpha}^0 - \frac{K_0}{2} (A_{0,0} + \partial A_{0,i} v^i) &= 0 \\ \ddot{\alpha}^i + \partial K A_{i,0} - \frac{K_0}{2} A_{0,i} + \\ &+ \partial K (A_{i,jk} - A_{k,j,i}) v^k + K \left(\frac{2\omega-2}{2\omega-3}\right) A_{0,0} v^i = 0 \end{aligned}$$

with

$$24) \quad A_{\mu, \alpha, \beta} \gamma^{\alpha\beta} = \rho_0 u_\mu \quad ; \quad u_\mu = (-1, 0, 0, v^i)$$

Thus in the case of the spherical shell of mass  $M$  radius  $R$ , rotating with small angular velocity  $\alpha$ , with respect to a system in which the metric is asymptotically Minkowskian, the "centrifugal force" induced on a test particle of mass  $m$  is the same as in the Einstein case, i. e.,  $\frac{MKm\alpha^2}{4\pi R}$ .

However, in terms of the observed  $K_0$  this is  $\frac{MK_0 m \alpha^2}{4\pi R} \left( \frac{2\omega-3}{2\omega-4} \right)$

Thus even in first approximation the extended set yields a relation between "induced inertial" and gravitational effects quantitatively different from the Einstein set. However, as in the Einstein case, it is obviously inadequate for consideration of realistic models, since it assumes  $\frac{KM}{R} \ll 1$ .

### VIII. Jordan's Work

#### A. Introduction: Field Equations;

#### Pauli Transformation

Jordan<sup>31</sup> and his coworkers have done a considerable amount of work on modifications of Einstein theory very similar to that discussed in section VII, pages 48 f.

His book contains, in addition to standard Einstein theory, an exposition of a five dimensional projective "unified field theory" which offers a variational principle containing a

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31 P. Jordan, Ibid.

"varying gravitational constant" to be determined by the field equations. After briefly discussing this Jordan restricts himself to the four dimensional form of the variational principle. The main form he uses in his book for this is<sup>32</sup>

$$1) \quad \delta \int d^4x \sqrt{-g} \left( R + \sum \frac{\chi_{,\alpha} \chi^{,\alpha}}{\chi^2} + \frac{\chi}{\partial} F_{\alpha\beta} F^{\alpha\beta} \right) = 0$$

leading to the field equations<sup>33</sup>

$$2) \quad R + \sum \left( -\partial \frac{\chi^{,\alpha}}{\chi} + \frac{\chi^{,\alpha} \chi_{,\alpha}}{\chi^2} \right) = \chi (F_{\alpha\beta} F^{\alpha\beta})$$

$$3) \quad -R_{\alpha\beta} + \frac{1}{\partial} g_{\alpha\beta} R + \frac{\chi_{,\alpha;\beta}}{\chi} - g_{\alpha\beta} \frac{\chi^{,\mu}{}_{;\mu}}{\chi} \\ - \sum \left( \frac{\chi_{,\alpha} \chi_{,\beta}}{\chi^2} - \frac{1}{\partial} g_{\alpha\beta} \frac{\chi^{,\mu} \chi_{,\mu}}{\chi^2} \right) = \chi (E_{\alpha\beta})$$

$$E_{\alpha\beta} \equiv F_{\alpha\mu} F_{\beta}{}^{\mu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}$$

He inserts other "matter" in the same way as the electromagnetic stress tensor  $E_{\alpha\beta}$  enters but claims that he need consider only the case where

32. P. Jordan, Ibid., p. 164. Here and in the following Jordan's notation and metric signature will be replaced by those chosen in this paper.

33. P. Jordan, Ibid., p. 165.

the time average of the Lagrange density for matter (corresponding to  $F_{\mu\nu} F^{\mu\nu}$ ), vanishes. This is indeed satisfied for the case of incoherent waves.<sup>34</sup> With this condition, then, he writes for the field equations in the presence of matter<sup>35</sup>

$$4) \quad -R + \Sigma \left( \frac{2\chi'^{\mu\nu}}{\chi} - \frac{\chi'^{\mu\nu}\chi_{,\mu\nu}}{\chi^2} \right) = 0$$

$$5) \quad -R_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}R + \frac{\chi_{,\alpha;\beta}}{\chi} - g_{\alpha\beta} \frac{\chi'^{\mu\nu}}{\chi} - \Sigma \left( \frac{\chi_{,\alpha}\chi_{,\beta}}{\chi^2} - \frac{1}{2}g_{\alpha\beta} \frac{\chi'^{\mu\nu}\chi_{,\mu\nu}}{\chi^2} \right) = -\chi T_{\alpha\beta}$$

where  $T_{\alpha\beta}$  is the "total energy tensor," radiation plus matter.

He then remarks that if  $T^{\alpha}_{\alpha} = 0$ , 4) and 5) are formally invariant under a "Pauli conformal transformation." Specifically, if  $\chi$  is a constant not equal to one, the replacements

$$g_{\alpha\beta} \rightarrow g_{\alpha\beta}^* = \chi^{\gamma} g_{\alpha\beta}$$

$$\chi \rightarrow \chi^* = \chi^{1-\gamma}$$

$$6) \quad \Sigma \rightarrow \Sigma^* = \frac{\Sigma}{\chi} + \frac{1}{(1-\gamma)^2} \left( \Sigma - \frac{\Sigma}{\chi} \right)$$

$$T_{\alpha\beta} \rightarrow T_{\alpha\beta}^* = \chi^{\gamma} T_{\alpha\beta}$$

leave 4) and 5) formally unchanged, provided  $T^{\alpha}_{\alpha} = 0$ .

34 P. Jordan, Ibid., p. 96,

35 P. Jordan, Ibid., P. 165.

### B. The Meaning of $\kappa$ and $T_{\alpha\beta}$

From 5), it might indeed seem that  $\kappa$  plays the role of gravitational constant," since it multiplies the "total energy tensor,"  $T_{\alpha\beta}$ . However,  $T_{\alpha\beta}$  is not conserved in general. In fact,<sup>36</sup>

$$7) \quad (\kappa^2 T^{\alpha\beta})_{;\beta} = 0$$

Later, he gives some hint of his interpretation of  $T_{\alpha\beta}$  by making the "Eddington-Pauli" postulate: "That tensor  $\kappa T_{\alpha\beta}$  for which the conservation law  $(\kappa T^{\alpha\beta})_{;\beta} = 0$  is valid, is to be interpreted as the matter tensor."<sup>37</sup> In the light of this 7) would require that  $\kappa^2 T^{\alpha\beta}$  be taken as the "matter tensor." Hence, on the right side of 5) the "matter tensor" is multiplied by  $\kappa^{-1}$  instead of  $\kappa$ , i. e.,  $\kappa^{-1}$ , not  $\kappa$ , plays the role of the "gravitational constant," Jordan does not discuss this problem in detail in his book although he does seem to be aware of it.<sup>38</sup>

### C. The Heckmann Solution;

#### Three Standard Tests

He then proceeds to derive and investigate the "Heckmann solution," the static spherically symmetric vacuum ( $T_{\alpha\beta} = 0$ )

36 P. Jordan, Ibid., p. 167.

37 P. Jordan, Ibid., My translation of a paragraph on page 171.

38 P. Jordan, Ibid., p. 172.

solution to 4) and 5) in Schwarzschild type coordinates. The form of this solution is quite complicated and only a parametric representation of it can be given in terms of elementary functions. That is, the coordinate radius and metric components are given as functions of the same parameter. This solution has essentially three free constants,  $\beta_0, \lambda_0, \chi_0$ . For certain ranges of these constants this solution behaves analogously to the Schwarzschild solution for large  $\lambda$  with  $\chi_0$  being the asymptotic value of  $\chi$ . He introduces a "gravitational radius"  $m$  such that

$$8) \quad g_{00} \xrightarrow{\lambda \rightarrow \infty} 1 - \frac{2m}{\lambda}$$

A comparison of this with the corresponding expansion of the Heckmann solution places one condition  $\beta_0$  and  $\lambda_0$  in terms of  $m$ . Noticing that 4) and 5) require

$$9) \quad \square \chi = \frac{\chi^{2.7}}{3-2.5}$$

he asserts that in the weak field approximation

$$10) \quad \chi \approx \chi_0 \left( 1 - \frac{2m}{(2.5-3)\lambda} \right)$$

Apparently he assumes that this "m" is the same as that introduced in (and defined by) 8). However, it is not at all obvious that the field equations, 4) and 5) will imply that the  $m$  appearing in 10), (which is really  $-\frac{\chi_0}{8\pi} \int \nabla d^3 \chi$ ) will be the same as the gravitational radius defined in 8). Nevertheless, he compares 10) with the corresponding expansion of the Heckmann solution to evaluate  $\beta_0$  in terms of "m".

Using this evaluation of  $\beta_0$  and  $\lambda_0$  he proceeds to investigate the three standard tests: red shift, light deflection, and perihelion rotation. He finds that the Einstein results are approached for large  $|\xi|$ .

#### D. Jordan's Cosmology

He then investigates the analogue of the Friedmann cosmological solution. He uses for the metric and "matter tensor"<sup>39</sup>

$$ds^2 = -dt^2 + (R(t))^2 d\sigma^2$$

$$T_0^0 = -\rho$$

$$11) \quad T_0^i = 0 \quad ; i \neq 0$$

$$T_i^j = p \delta_i^j$$

$$\rho = \rho(t) \quad ; p = p(t) \quad ; \chi = \chi(t)$$

where  $d\sigma^2$  represents the usual space metric of curvature  $\pm 1$  or 0. He presents an analysis<sup>40</sup> (including numerical calculations) of some of the mathematics involved in solving the field equations for such a universe but does not clarify the physical interpretation of either  $\chi$  or the "mass density"  $\rho$ . Most of the discussion is restricted to positive  $\chi$ . This is due to the conjecture

39 P. Jordan, Ibid., p. 186.

40 P. Jordan, Ibid., pp. 196-207.

$$12) \quad \frac{1}{\chi} = \frac{1}{\chi_0} \left( 1 + A \frac{m}{\lambda} \right) \quad ; \quad A > 0$$

and the form 10) of the weak field solution. If the space metric corresponds to positive unit curvature, he claims<sup>41</sup> that if  $\lambda > 0$ , as  $t \rightarrow \infty$  all solutions with  $P=0$  approach a special linear solution

$$R = \left( \frac{2}{\lambda - 2} \right)^{\frac{1}{2}} t$$

$$13) \quad \chi = \frac{\chi_0}{t}$$

$$\rho = \frac{2\lambda - 3}{\chi_0 t}$$

$$\chi_0 = \text{constant}$$

This solution seems to fit Dirac's  $K \sim \frac{1}{t}$  conjecture. Unfortunately, however, it does not seem that Jordan's  $\chi$  would appear as the real locally measured Newtonian gravitational constant.<sup>42</sup>

He also includes a study of the non-static spherically symmetric vacuum solution since the Birkhoff theorem is not applicable to 4) and 5).

The book concludes with a rather extended discussion of some of the possible geological and astronomical consequences of the theory.

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41 P. Jordan, Ibid., p. 200.

42 See the discussion in VIII-B, page 60 above as well as that of Fierz VIII-E, page 64f.

### E. Fierz's Critique

Fierz<sup>43</sup> makes use of the invariance of the field equations under the Pauli conformal transformation to point out some physical ambiguities in Jordan's results. He asserts that it is necessary to explicitly put a "matter" term in the variational principle in such a way as to give a "matter tensor" having non-zero trace. This then destroys invariance under the Pauli transformation.

Fierz writes Jordan's action as

$$14) \quad \int d^4x \sqrt{-g} \left\{ \chi \left[ R + \frac{\chi_{,a} \chi^{,a}}{\chi^2} \right] + \frac{\epsilon_0}{2} F_{kl} F^{kl} \right\}$$

where  $\epsilon_0 \equiv \chi^{1+\frac{1}{n}}$ ,  $n = \text{constant}$ ,

To this he adds the matter action

$$15) \quad m \int h(\chi) \sqrt{g_{\mu\nu} \frac{\chi^{,\mu} \chi^{,\nu}}{\chi^2}} d\lambda$$

where  $h$  is an arbitrary function of  $\chi$ . He then makes the important postulate that a mass point should follow a geodesic. For arbitrary  $h$  and variable  $\chi$  this can not be satisfied if the  $g_{\mu\nu}$  in 15) is interpreted as the observed metric. Rather,  $\bar{g}_{\mu\nu} \equiv (h(\chi))^2 g_{\mu\nu}$  must be taken as the meaningful metric. Putting this into the remaining action for the gravitational, electromagnetic and  $\chi$  fields gives a slightly modified action in which  $\frac{(h(\chi))^2}{\chi}$  appears in the role of gravitational "constant." Fierz notes only two cases. If  $h(\chi) = \chi^{\frac{1}{2}}$

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43 M. Fierz, Ibid. Jordan has taken note of Fierz's paper. See Z. Phys., 157 112 (1959).

standard Einstein theory results. If  $h(\chi) = 1$ ,  $\chi^{-1}$  seems to be the gravitational "constant." The resultant total action is

$$16) \quad \int \left( \frac{R}{\chi} + \sum \frac{\chi_{,\mu} \chi^{,\mu}}{\chi^3} + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right) F g d^4x + \\ + m \int \sqrt{g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}} d\lambda$$

Fierz shows the same results hold for a wave mechanical action for matter.

#### F. Extension and Amplification of Jordan Theory

Others have investigated some of the questions associated with Jordan's exposition. In particular, Just<sup>44</sup> has added a matter Lagrangian, calculated the corresponding stress tensor to be added to the field equations, and expressed the constants in the Heckmann solution in terms of space integrals over the masses of functions containing components of their stress tensor, metric tensor, and  $\chi$ . He then calculates the equations of planetary motion in terms of these constants. Later<sup>45</sup> he considers cosmology in view of this work. Ludwig and Just<sup>46</sup> analyze the resulting equations of motion, which do not describe a geodesic. This is due to the fact that what they call the "matter" tensor is not conserved. This they regard as favorable and discard a variational principle like 16) or VII-6) page 51 since it leads to no "entstehung von neuer Materie."<sup>47</sup> Just<sup>48</sup> further discusses

44 K. Just, Z. Phys., 140 524 (1955).

45 K. Just, Z. Phys., 140 648 (1955). Z. Phys., 141 592 (1955).

46 G. Ludwig and K. Just, Z. Phys., 143 472 (1955).

47 G. Ludwig and K. Just, Ibid.

48 K. Just, Z. Phys., 144 411 (1956), 144 648 (1956).

the evaluation of the constants in the Heckmann solution in terms of the "matter" tensor, and in what ranges the available parameters must lie in order not to contradict observations such as planetary motions and Eötvös experiment.

#### G. Comparison of Work of This Paper to That of Jordan

In this paper, the function of interest,  $\phi$ , corresponds to the reciprocal of the gravitational "constant" rather than the gravitational "constant" itself as chosen by Jordan. Reasons for this choice were set forth in VI-B, page 36 f. It is  $\phi$  which has units of  $ML^{-1}$  and might be expected to satisfy an inhomogeneous wave equation with matter as source,

We explicitly put matter terms in the variational principle and field equations. Further, this is done in such a way as to leave the actions associated with matter and electromagnetic fields unchanged so that the equations satisfied by the matter variables are formally identical with those in standard general relativity. In particular, the matter tensor is conserved and a test particle follows a geodesic. Hence, while the strong principle of equivalence is violated, the weak principle is exactly satisfied so that, for example, a null result is predicted in the Eötvös experiment. Further standard electromagnetic theory is left unchanged so that the inertial determination of mass in charge-time units is identical with that in general relativity.<sup>49</sup> This is done for the Infeld source tensor used in determining equations of motion in IX-C pages 71 f. below. In other words, an attempt is made to associate a physically observable number, inertial mass, with the formal "stress tensor of

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<sup>49</sup> See IV-E, pages 21 f.

matter" put into the field equations. The method chosen, based on conserved electric charge and constant dielectric "constant," might be open to discussion,<sup>50</sup> but is sufficient for the purposes of this paper.

Further in this paper an attempt is made to relate  $\frac{1}{\phi}$  to the actually measured Newtonian gravitational constant (see X pages 74 f.). It is not sufficient to simply say that since  $\frac{1}{\phi}$  enters the field equations analogously to  $K$  in Einstein equations it is to be identified with the locally measured gravitational constant.

A comparison of the vacuum field equations VII-8) and 9) page 51 with those of Jordan 4) and 5) show that they are formally identical if  $\xi = \omega$  and  $\phi$  is replaced by  $\chi$ . Hence the solutions, while formally identical, are physically very different unless Jordan's  $\chi$  is taken to be the reciprocal of the gravitational constant. In this paper the solution to VII-8) and 9) page 51 in the time-independent spherically symmetric vacuum case will be given in isotropic coordinates (See XII pages 79f). Because of the formal identity of VII-8) and 9) page 51 with 4) and 5) the geometry of this solution is the same as that for the Heckmann solution but in isotropic coordinates the metric components and  $\phi$  can be directly given in terms of elementary functions of the coordinate radius. This solution is then applied to the problem of boundary conditions on  $\phi$  (See XIII pages 87 f.).

The cosmological equations considered in this paper are again formally identical with those of Jordan's. However, the replacements of  $\phi$  by  $\chi$  and  $\rho$  by  $\chi^2 \rho$  necessary to achieve this radically alter the physical meaning and interpretation of results. Further, Jordan is largely concerned with solutions associated with

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50 In addition to Jordan's work see also R. H. Dicke, Rev. Mod. Phys., 29 363 (1957), Science, 129 621 (1959), Am. J. Phys., 28 344 (1960).

ranges of  $\Sigma$  (or  $\omega$ ) greater than zero since he wants  $\chi^{-1}$  to be a positive sum of terms like  $\frac{m}{r}$ . On the other hand, in this paper this is a property of  $\phi$  (which is his  $\chi^{+1}$ ) so that  $\omega < 0$  is the range of most probable interest here.

Also, in addition to several exact special solutions, this paper contains the first few terms of an expansion of the cosmological solution and associated quantities in powers of  $\frac{1}{\omega}$ , XIV pages 100 f. This seems appropriate since, to fit observations, the theory must give predictions fairly close to those of the Einstein theory. This requires large  $\omega$ , e.g.  $|\omega| \geq 10$  (See VII-D pages 53 f.)

## IX. Infeld's Approximation Procedure

### A. Introduction: Assumptions

In order to obtain a better picture of how the strong principle of equivalence is violated by the introduction of  $\phi$ , consider the many body problem through second approximation. This is required<sup>51</sup> to obtain the effect of matter on the physically measured gravitational constant as defined in X page 74f. below. The procedure adopted is that of Infeld<sup>52</sup>, neglecting internal particle structure.

It is obviously sufficient to assume the "zerth" approximation of  $\phi$  as the reciprocal of a constant of the same order of magnitude as the usual gravitational constant in order to stay "near" an Einstein solution. Hence, set

$$\phi = \frac{1}{K} (1 + {}_2\psi + {}_4\psi)$$

$$1) \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

<sup>51</sup> Cf. Part One II-C, pages 8 f.

<sup>52</sup> L. Infeld Rev. Mod. Phys., 29, page 398 (1957).

with the  $h_{\mu\nu}$  expanded

$$h_{00} = {}_2h_{00} + {}_4h_{00} + \dots$$

2)

$$h_{0i} = {}_3h_{0i} + {}_5h_{0i} + \dots$$

$$h_{ij} = {}_2h_{ij} + {}_4h_{ij} + \dots$$

and assume time differentiation to increase order by one.

### B. Infeld's $\delta$ Function

The choice of the matter tensor in this method is a special type of  $\delta$  function which is assumed to be already renormalized to contain all self interaction effects for spherically symmetric point particles. More precisely, if the path of the particle is  $x^\mu = \xi^\mu(\lambda)$  for some parameter  $\lambda$ , and if  $f$  is some quantity expandable about  $x^\mu = \xi^\mu(\lambda)$  as

$$3) \quad f(x) = \overline{f(\lambda)} + \sum_{n=-k}^{-1} \overline{f^{(n)}(\lambda)} (\lambda)^n + \sum_{n=1}^{\infty} \overline{f^{(n)}(\lambda)} (x^{i_1} - \xi^{i_1}) \dots (x^{i_n} - \xi^{i_n})$$

where

$$4) \quad h(\lambda) \equiv \left[ \sum_{i=1}^3 (x^i - \xi^i(\lambda))^2 \right]^{\frac{1}{2}}$$

then the matter tensor  $T^{\alpha\beta}$  is assumed to have the following properties

$$5) \quad T^{\alpha\beta}(x) = 0 \quad \text{for } x \notin \{x; x = \xi(\lambda) \text{ for some } \lambda\}$$

and

$$6) \int_{\sigma} f(x) T^{\alpha\beta}(x) * \alpha = \frac{N(\lambda_0) \sqrt{f(\lambda_0)} \mu(\lambda_0) \xi^{\mu}(\lambda_0) \xi^{\nu}(\lambda_0) (\alpha \cdot \eta)(\lambda_0)}{(\eta \cdot \xi)(\lambda_0)}$$

where  $\alpha$  is any one form,  $\sigma$  is any space-like surface of normal  $n$  intersecting the path  $x^{\mu} = \xi^{\mu}(\lambda)$  only at  $\xi(\lambda_0)$ ,  $\xi^{\alpha} \equiv \frac{d\xi^{\alpha}}{d\lambda}$ ,  $\mu(\lambda_0)$  is a scalar and  $N(\lambda_0)$  a normalization factor. If  $\sigma$  does not intersect  $\{x = \xi(\lambda)\}$ , the right hand side would be zero. This is simply an invariant formulation of Infeld's definition, showing immediately that  $T^{\alpha\beta}$  must transform as a tensor. For  $f = g_{\mu\nu}$ , the proper time along the path is defined by

$$7) d\tau = \left( -g_{\mu\nu}(\lambda) \xi^{\mu} \xi^{\nu} \right)^{\frac{1}{2}} d\lambda$$

Assume then that  $N(\gamma) = 1$  for all  $\gamma$ . From the conservation equations  $T^{\alpha\beta}_{;B} = 0$  it is a straightforward matter to verify

$$8) \left( \mu \xi^{\alpha} \right)_{;B} \xi^B = 0$$

(covariant differentiation with respect to background metric). Thus, assuming  $\overline{g_{\mu\nu}} = \overline{g_{\mu\nu}} = \{ \mu, \nu, \alpha \}$ ,  $\dot{\mu} = 0$  and  $\mu$  is a constant of the motion for a free particle.

As discussed in the introduction, VI-D, page 39f., use of Infeld's  $\delta$  function can be regarded mainly as a computational convenience. Of course, it does entail an assumption that the masses have an overall point particle like behavior. There is every reason

to believe that the Papapetrou-Fock method, corresponding to extended "fluid" bodies, would give the same result for the equation of motion of centers of "mass" as the infeld procedure. In fact, these two methods do give the same observable results in the Einstein case. The introduction of  $\phi$  changes this by requiring an evaluation of the integral of the trace of the stress tensor of the mass. This number is the same for both methods,  $m$  for Papapetrou-Fock (IV-D eq. 24 page 21) and  $\mathcal{M}$  for Infeld. Both  $m$  and  $\mathcal{M}$  turn out to be equal to the observed inertial mass in charge units. (IV-E page 21 f., and IX-C pages 71 f.)

### C. Relation of $\mu$ to Inertial Mass

In order to relate  $\mathcal{M}$  to an inertial mass, assume that the application of a charge  $e$  to the particle is represented by a current density  $j^\alpha(x)$

$$9) \quad j^\alpha(x) = 0 \quad \text{for } x \notin \{x: x = F(\lambda) \text{ for some } \lambda\}$$

and

$$10) \quad \int_{\sigma} f j^\alpha * \beta = e \overline{f(\lambda_0)} \xi'^\alpha \left( \frac{(\beta, \eta)(\lambda_0)}{(\eta, \xi)(\lambda_0)} \right)$$

Thus the total flux produced by the particle is

$$11) \quad \int_{x^0 = \text{constant}} j^0 \sqrt{-g} d^3x = e$$

In the presence of a field  $F^{\alpha\beta}$ , conservation of total stress tensor yields

$$12) \quad T^{\alpha\beta}_{;\beta} = j^{\beta} F^{\alpha}_{\beta}$$

On integrating both sides over  $\chi^0 = \text{constant}$  this gives

$$13) \quad (\mu \xi'^{\alpha})_{;\beta} \xi'^{\beta} = e \overline{F^{\alpha}_{\beta}} \xi'^{\beta}$$

Hence  $\mu$  must be identified with the observed inertial mass since all the fields,  $g_{\mu\nu}$  and  $F_{\mu\nu}$  occurring in the equation are "external."

#### D. Evaluation of Metric and Equations of Motion

Let the particles be labeled by a, b, c, . . . and their positions  $a^i, b^i, c^i, \dots$ , ( $i = 1, 2, 3; \frac{dx^i}{dt} \equiv \dot{x}^i$ )

with

$$11) \quad \Lambda_a \equiv \left[ \sum_{i=1}^3 (x^i - a^i)^2 \right]^{\frac{1}{2}} ; \Lambda_{ab} \equiv \left[ \sum_{i=1}^3 (a^i - b^i)^2 \right]^{\frac{1}{2}} ; v_a^2 \equiv \sum_{i=1}^3 (\dot{a}^i)^2$$

The following results are found

$$12) \quad \begin{aligned} {}_2 h_{00} &= -f & ; {}_2 h_{ij} &= -\delta_{ij} f \left( \frac{\omega-1}{\omega-2} \right) \\ {}_2 \psi &= \frac{f}{2\omega-4} & ; {}_3 h_{0i} &= 2 \sum_a \frac{f_a}{a} \dot{a}^i \left( \frac{2\omega-3}{2\omega-4} \right) \\ {}_4 h_{00} &= -\frac{f^2}{2} - \sum_a \left( \frac{f_a h_0^2}{2} \right)_{,0,0} - \frac{1}{2\omega-4} \sum_a \frac{f}{a} \left[ v_a^2 (3\omega-4) + f(\omega-2) \right] \end{aligned}$$

where

$$f \equiv \sum_a f_a \quad ; \quad \overset{a}{f} \equiv \sum_{b \neq a} f_b(a)$$

13)

$$f_a(x) \equiv -2 \frac{K_0 \mu_a}{8\pi \lambda_a}$$

As a check it is observed that in the limit  $\omega \rightarrow \infty$ , these solutions approach those given by Infeld. After lengthy but straightforward calculation the coordinate acceleration of the  $a^{\text{th}}$  particle is found to be given in the expression

$$\begin{aligned} \ddot{a}^i &= \sum_{b \neq a} M_b \left\{ \frac{\partial}{\partial a^i} \frac{1}{\lambda_{ab}} + \left[ v_a^2 \frac{(\omega-1)}{(\omega-2)} + v_b^2 \frac{(3\omega-4)}{2(\omega-4)} \right. \right. \\ &\quad \left. \left. - 4 \dot{a}^j \dot{b}^j \frac{(\omega-3)}{2(\omega-4)} - \frac{2M_b}{\lambda_{ab}} \frac{(\omega-3)}{(\omega-2)} - \frac{M_b}{\lambda_{ab}} \frac{(5\omega-8)}{(\omega-2)} \right] \frac{\partial}{\partial a^i} \frac{1}{\lambda_{ab}} + \right. \\ &\quad \left. + \left[ -4 \dot{b}^i \dot{b}^j \frac{(\omega-3)}{2(\omega-4)} + 4 \dot{a}^j \dot{b}^j \frac{(\omega-3)}{2(\omega-4)} + \dot{a}^i \dot{b}^j \frac{(3\omega-4)}{(\omega-2)} \right. \right. \end{aligned}$$

14)

$$\begin{aligned} &\left. - 2 \dot{a}^i \dot{a}^j \frac{(\omega-3)}{(\omega-2)} \right] \frac{\partial}{\partial a^k} \frac{1}{\lambda_{ab}} + \frac{1}{2} \dot{b}^j \dot{b}^k \frac{\partial^3 \lambda_{ab}}{\partial a^i \partial a^j \partial a^k} \\ &+ \frac{1}{2} \sum_{\substack{c \neq b \\ c \neq a}} M_c \left[ \frac{1}{\lambda_{ab}} \frac{\partial}{\partial b^c} \frac{1}{\lambda_{bc}} \frac{(\omega-10)}{(\omega-2)} - \frac{2}{\lambda_{bc}} \frac{\partial}{\partial a^i} \frac{1}{\lambda_{ab}} \right. \\ &\left. - 4 \frac{(\omega-3)}{(\omega-2)} \frac{1}{\lambda_{ac}} \frac{\partial}{\partial a^i} \frac{1}{\lambda_{ab}} + (b^d - a^d) \left( \frac{\partial}{\partial a^i} \frac{1}{\lambda_{ab}} \right) \left( \frac{\partial}{\partial b^d} \frac{1}{\lambda_{bc}} \right) \right] \end{aligned}$$

where

$$M_b \equiv \frac{K_0 \mu_b}{8\pi}$$

## X. Definition of Local Gravitational "Constant"

### A. Introduction: Definition of $K_E$

The definition of local gravitational "constant" will be based on the comparison of the relative motion of two small bodies to that predicted by a Newtonian theory of gravity. As pointed out in III pages 10f., motion must be invariantly described to correspond to a real experimental result.

Consider now the measurement of the local effective gravitational constant  $K_E$  defined as follows.

$$1) \quad K_E \equiv -8\pi \lim_{r_p \rightarrow 0} r_p^2 \left( \frac{\partial A_p}{\partial \mu} \right)_{\mu=0}$$

Here  $A_p$  is the proper relative radical acceleration  $\left( \frac{d^2}{dt^2} r_p \right)$  of a test particle instantaneously at rest at a proper distance  $r_p$  (along  $dt=0$ ) from a spherically symmetric inertial mass  $\mu$  in a local time orthogonal coordinate system. Of course, in defining proper distances, the singularity must be replaced by an extended mass. The above definition is also seen to be equivalent to that in which  $r_p$  is defined as one half the proper time (on the test particle) of flight of a light ray to the gravitating mass and back with no coordinate conditions.

This definition of  $K_E$  is chosen for the following reasons. Since  $(A_p)_{\mu=0}$  may not be zero (the entire coordinate system may be accelerating) and only the change in  $A_p$  due to  $\mu$  is desired,  $\frac{\partial A_p}{\partial \mu}$  is used. Also, it is evaluated at  $\mu=0$  since only the Newtonian limit is desired and spurious effects due to the interaction of  $\mu$  with background matter are to be eliminated. The limit,  $r_p \rightarrow 0$ , is chosen to eliminate background curvature effects.

B. Evaluation of  $K_E$  from Equations of Motion: An Example

For  $M_b$  instantaneously at rest at the origin with the rest of the universe at rest and  $\frac{M}{R} \equiv \sum_{c \neq a} \frac{M_c}{R_{ac}}$ , the acceleration of a test particle to first order in  $M_b$  is from IX-14) page 73

$$2) \quad \ddot{a}^i = \frac{K_0 M_b}{8\pi R} \left( 1 - \frac{5\omega - 8}{\omega - 2} \frac{K_0 M}{8\pi R} \right) \frac{\partial}{\partial a^i} \frac{1}{R_{ab}} + \text{order} \left( \frac{1}{R_{ab}^2}, R_{ab}'' \dots \right)$$

where

$$3) \quad K = \lim_{R \rightarrow \infty} \frac{1}{\phi(R)} \quad ; \quad K_0 \equiv \frac{2\omega - 4}{2\omega - 3} K$$

Hence<sup>53</sup>

$$4) \quad K_E = \left( 1 + \frac{K_0 M}{8\pi R(\omega - 2)} \right) K_0$$

Clearly as  $\omega \rightarrow \infty$ ,  $K_E \rightarrow K_0 \rightarrow K$ , independent of  $M$ ,  $R$  as in the Einstein theory. This might have been anticipated from first order approximation theory. The effect of  $M$  in local calculations is to replace the boundary value  $\frac{1}{R}$  by

$$5) \quad \frac{1}{R} (1 + M\psi)$$

Where from VII-11a) page 55,  $\frac{\psi}{M} = \frac{-K_0 M}{8\pi R(\omega - 2)}$ . For finite  $\omega$ ,

<sup>53</sup> See III-C, pages 12 f.

however, this result plainly demonstrates the violation of the strong principle of equivalence: the effect of the rest of the universe on local, proper, gravitational experiments cannot even approximately be "transformed" away. Further, this result is independent of the velocities of  $M_b$ , and the individual particles in the rest of the universe. In fact, a glance at IX-14) page 73 shows that the only velocity terms contributing to  $\ddot{a}'^i$  are those involving the velocities of  $M_b$  and the test particle, and further, since these are to be instantaneously equal, are independent of  $\omega$ . A direct calculation shows that they are precisely the terms that would appear in a Lorentz transformation (after transforming the background metric to the Minkowskian) of the relations between acceleration and distance for both gravitating and test particle instantaneously at rest. In particular, if all the velocities are the same, this shows that  $K_E$  as defined above is Lorentz invariant through order  $v^2$  and  $\frac{KM}{R}$ .

As an example, the maximum variation of  $K_E$  measured on earth due to its varying distance from the sun would be of order

$$6) \quad \frac{K_E^{\min}}{K_E^{\max}} \approx 1 + \frac{K_0 M}{9TT(\omega^2)} \left( \frac{1}{R^{\min}} - \frac{1}{R^{\max}} \right) \approx 1 + \frac{6 \times 10^{-8}}{\omega^2}$$

## XI. The Heckmann Solution and Three Standard Tests

### A. Introduction: Statement of Solution

The analogue of the Schwarzschild exterior solution has been given by Heckmann et. al.<sup>54</sup> for the equations VII-8) and 9) page 51. Their equations are formally identical with this set in the case  $\Gamma_{\alpha\beta}^{\gamma} = 0$

With

$$1) \quad ds^2 = -e^{2\mu} dt^2 + e^{2\nu} ds^2 + s^2 d\Omega^2$$

the results are

$$2) \quad \lambda = \frac{\lambda_0}{\sqrt{\gamma}(\gamma^{-h} - \gamma^h)} \quad ; \quad e^{2\mu} = \gamma^{\frac{1}{B}}$$

$$e^{2\nu} = \frac{4h^2}{\left[\left(\frac{1}{2}+h\right)\gamma^h - \left(\frac{1}{2}-h\right)\gamma^{-h}\right]^2}$$

$$\phi = \phi_0 \gamma^{\beta_0/B}$$

with  $\gamma$  a free parameter,

$$3) \quad B \equiv 1 + 2\beta_0 \quad ; \quad h^2 \equiv \frac{1}{4} - \frac{\beta_0(1 + \beta_0 \omega)}{2B^2}$$

and  $\phi_0, \beta_0, \lambda_0$  independent constants. A thorough examination of these functions can be found in Jordan<sup>55</sup>, and only a few illustrative examples considered here. Since  $\lambda \rightarrow \infty$  for either  $\gamma \rightarrow (1)^{\frac{1}{2h}}$  or  $\gamma \rightarrow 0$ ,  $e^{2\mu} \xrightarrow{\lambda \rightarrow \infty}$  either  $(1)^{\frac{1}{2h}}$  or 0.

54 P. Jordan, Ibid.

55 P. Jordan, Ibid.

B. Approximate Evaluation of Constants and Three Standard Tests

Restricting  $\gamma$  to the range  $K\gamma < \infty$ , for large  $\Lambda$ , a power expansion in terms of  $\gamma^{-1}$  gives to first order in  $\gamma^{-1}$  the metric

$$4) \quad e^{2\mu} \approx 1 - \frac{\lambda_0}{2hB\lambda} ; \quad e^{2\nu} \approx 1 + \frac{\lambda_0}{2h\lambda} ; \quad \phi \approx \phi_0 \left( 1 - \frac{\beta_0 \lambda_0}{2hB\lambda} \right)$$

Hence, comparison with the approximate solutions to VII-8a), 11a) gives

$$5) \quad \frac{\lambda_0}{2hB} \approx \frac{K_0 m}{4\pi} ; \quad \phi_0 \approx \frac{1}{K}$$

$$\frac{\beta_0 \lambda_0}{2hB} \approx \frac{K_0 m}{(2\omega-4)4\pi} ; \quad \frac{\lambda_0}{2h} \approx \frac{K_0 m (\omega-1)}{4\pi (\omega-2)}$$

Thus the deflection of a light ray (null geodesic) passing at distance  $\lambda$  from mass  $m$  is  $\frac{K_0 m}{2\pi\lambda} \left( 1 + \frac{1}{2\omega-4} \right)$ . The fractional difference with respect to the Einstein value is  $\frac{1}{2\omega-4}$ . Since the  $g_{00}$  component of the metric tensor is the same as in the Einstein case through first approximation, the gravitational red shift is approximately unchanged. Further, the approximate path of a planet about mass  $m$  is given by

$$6) \quad \frac{1}{\lambda} = \frac{K_0 m}{8\pi P^2} \left\{ 1 + e \cos \left[ \theta \left( 1 - 3 \left( \frac{K_0 m}{8\pi P} \right)^2 \frac{(3\omega-4)}{(3\omega-6)} \right) \right] \right\}$$

where  $P \equiv \lambda^2 \dot{\theta} = \text{constant}$ . Hence the rotation of the orbit per period is  $3 \left( \frac{K_0 m}{8\pi P} \right)^2 \left( 1 + \frac{2}{3\omega-6} \right)$ , the fractional difference with respect to the Einstein value being  $\frac{2}{3\omega-6}$ . This demonstrates the approach to the Einstein results for large  $\omega$ .

If  $e^{2\mu}$  approaches zero for  $\lambda \rightarrow \infty$ , that is,  $\gamma \rightarrow 0$ , clearly the range  $0 < \gamma < 1$  alone is inappropriate for  $|\lambda| < \frac{1}{2}$  since it

excludes the region  $\Lambda < \Lambda_0$  and covers the remaining space twice. For negative  $\gamma$ , if  $(-1)^{\frac{h}{b}} = 1$  then the range  $-\infty < \gamma < -1$  gives a repeat of the values for  $1 < \gamma < \infty$  if  $(-1)^{\frac{h}{b}}$  is real, otherwise  $e^{2\gamma}$  will be complex. If  $(-1)^{\frac{2h}{b}} \neq 1$ , in general  $\Lambda$  will be complex for negative  $\gamma$ .

## XII. Exact Spherically Symmetric Static Vacuum Solution in Isotropic Coordinates

### A. Metric and Field Equations

The use of isotropic coordinates simplifies the metric, at least in the case  $\Gamma_{\alpha\beta} = 0$ .

Consider then,

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} (dr^2 + r^2 d\Omega^2)$$

1)

$$\alpha = \alpha(r)$$

$$\beta = \beta(r)$$

For

$$2) \quad \Gamma_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + P g_{\alpha\beta}$$

$$\text{with } \rho = \rho(r) ; p = p(r) ; u_\alpha u^\alpha = -1 ; u_i = 0 \text{ for } i \neq 0$$

the field equations become

$$3) \quad -\phi e^{-2\beta} \left[ \partial \beta'' + \frac{4\beta'}{r} + \beta'^2 \right] = -e^{-2\beta} \left( \phi \phi' + \frac{\omega \phi'^2}{\partial \phi} \right) + \rho + \frac{\rho - 3p}{2\omega - 3}$$

$$4) \quad \phi e^{-2\beta} \left[ \partial \alpha' \beta' + \beta'^2 + \frac{\partial(\alpha + \beta)}{r} \right] = e^{-2\beta} \left( \phi'' - \phi \beta' - \frac{\omega \phi'^2}{\partial \phi} \right) + p - \frac{\rho - 3p}{2\omega - 3}$$

$$5) \quad \phi e^{-2\beta} \left[ \alpha'' + \alpha'^2 + \frac{\alpha + \beta}{r} + \beta'' \right] = e^{-2\beta} \left[ \phi' \left( \beta' + \frac{1}{r} \right) + \frac{\omega \phi'^2}{\partial \phi} \right] + p - \frac{\rho - 3p}{2\omega - 3}$$

$$6) \quad e^{-2\beta} \left[ \dot{\phi} + (\alpha' + \beta' + \frac{2}{\lambda}) \phi \right] = \frac{\rho - 3p}{2\omega - 3}$$

$$7) \quad \dot{p} + (\rho + p) \alpha' = 0$$

$$\dot{f} \equiv \frac{df}{d\lambda}$$

### B. Differential Equations for the Vacuum: Type One Solution

For the exterior case  $\rho = p = 0$ , 6) yields

$$8) \quad \dot{\phi} = \frac{A e^{-\alpha - \beta}}{\lambda^2} \quad ; \quad A = \text{constant}$$

while the  $R_{00}$  equation becomes equivalent to

$$9) \quad \frac{\dot{\phi}}{\phi} + \alpha' + \beta' + \frac{2}{\lambda} = -\frac{\dot{\phi}}{\phi} \quad (\text{for } \alpha' \neq 0 \neq \phi)$$

Thus

$$10) \quad \alpha' \lambda^2 \phi e^{\alpha + \beta} = \frac{A}{C} \quad ; \quad C = \text{constant}$$

From 8) and 10) then

$$11) \quad \frac{\dot{\phi}}{\phi} = C \alpha' \quad ; \quad \phi = \phi_0 e^{C\alpha} \quad ; \quad \phi_0 = \text{constant}$$

Setting  $y \equiv \frac{d\beta}{d\alpha}$ , 3) becomes, using 9) and 11)

$$12) \quad \frac{dy}{d\alpha} = \frac{C(1 + \frac{\omega C}{2})}{2} + (C+1)y + \frac{1}{2}y^2$$

With

$$13) \quad \lambda^2 \equiv (C+1)^2 - C(1 + \frac{\omega C}{2})$$

12) becomes, for  $\lambda^2 > 0$ ,

$$14) \quad y = \frac{(\lambda + c + 1) e^{\lambda(\alpha - \alpha_0)} + \lambda - (c + 1)}{1 - e^{\lambda(\alpha - \alpha_0)}} ; \alpha_0 = \text{constant}$$

so that

$$15) \quad e^{\beta - \beta_0} = \frac{e^{[\lambda - (c + 1)](\alpha - \alpha_0)}}{(1 - e^{\lambda(\alpha - \alpha_0)})^2} ; \beta_0 = \text{constant}$$

Inserting this into 10) yields

$$16) \quad \frac{\lambda A e^{\lambda(\alpha - \alpha_0)}}{(1 - e^{\lambda(\alpha - \alpha_0)})^2} = -\frac{B}{\lambda^2} ; B \equiv \frac{\lambda A e^{-\beta_0 - (c + 1)\alpha_0}}{\phi_0 C}$$

so that

$$17) \quad \frac{1}{1 - e^{\lambda(\alpha - \alpha_0)}} = \frac{B}{\lambda^2} + D ; D = \text{constant}$$

Aside from the trivial constants,  $\alpha_0$ ,  $\beta_0$ , four independent constants  $\phi_0$ , A (or B), C, D have been introduced. The constraint equation, 4) in this case, may be expected to eliminate one of them. In fact, it becomes

$$18) \quad \frac{dA}{d\lambda} + \left( \frac{1 + y + c}{\lambda} \right) = 0$$

which requires  $D = \frac{1}{2}$ . Hence the final solution is of the form (re-defining constants  $\alpha_0$ ,  $\beta_0$ ,  $\phi_0$ )

$$\begin{aligned}
 & \text{I. } \left\{ \begin{array}{l}
 19) \quad e^\alpha = e^{\alpha_0} \left[ \frac{1 - \frac{2B}{\lambda}}{1 + \frac{2B}{\lambda}} \right]^{\frac{1}{\lambda}} \\
 20) \quad e^\beta = e^{\beta_0} \left(1 + \frac{2B}{\lambda}\right)^2 \left[ \frac{1 - \frac{2B}{\lambda}}{1 + \frac{2B}{\lambda}} \right]^{\frac{2-C-1}{\lambda}} \\
 21) \quad \phi = \phi_0 e^{\alpha_0 C} \left[ \frac{1 - \frac{2B}{\lambda}}{1 + \frac{2B}{\lambda}} \right]^{\frac{C}{\lambda}} \\
 \lambda \equiv \left[ (C+1)^2 - C \left(1 + \frac{\omega C}{3}\right) \right]^{\frac{1}{2}} \quad ; \quad \lambda^2 > 0
 \end{array} \right.
 \end{aligned}$$

### C. Approximate Evaluation of Constants

#### From Weak Field Solution

Setting  $\alpha_0 = \beta_0 = 0$ , an expansion in powers of  $B/r$  yields in Cartesian coordinates

$$g_{00} = \eta_{00} + \frac{8B}{\lambda r} + \dots \quad ; \quad g_{0i} = 0$$

$$22) \quad g_{ij} = \eta_{ij} \left(1 + \frac{8B(C+1)}{\lambda r}\right) + \dots$$

$$\phi = \phi_0 \left(1 - \frac{4BC}{\lambda r} + \dots\right)$$

Comparison with the weak field solutions yields

$$\phi_0 = \frac{1}{K} + \dots \quad ; \quad \frac{8B}{\lambda} = \frac{K_0 m}{4\pi} + \dots$$

$$23) \quad C = \frac{1}{\omega - 2} + \dots$$

D. Solutions of Types Two and Three

The case  $\lambda^2 < 0$  is easily considered by setting  $\lambda = \sqrt{-1} \Lambda$ , replacing  $\beta_0$  by  $\beta_0 + \sqrt{-1} \pi$ , leaving  $\alpha_0$  real and replacing B by  $\sqrt{-1} B$ . The net result is

$$\text{II. } \left\{ \begin{array}{l} 24) \quad \alpha - \alpha_0 = \frac{2}{\Lambda} \tan^{-1} \frac{\Delta}{2B} \\ 25) \quad \beta - \beta_0 = \frac{-2(c+1)}{\Lambda} \tan^{-1} \frac{\Delta}{2B} - \ln \frac{\Lambda^2}{\Lambda^2 + 4B^2} \\ 26) \quad \phi = \phi_0 e^{\alpha_0 c} e^{\frac{2c}{\Lambda} \tan^{-1} \frac{\Delta}{2B}} \\ \Lambda \equiv [c(1 + \frac{\omega c}{3}) - (c+1)^2]^{\frac{1}{2}} \quad ; \quad \Lambda^2 > 0 \end{array} \right.$$

Finally, the case  $\lambda^2 = 0$  requires,

$$27) \quad c = \frac{1 \pm \sqrt{2\omega - 3}}{\omega - 2}$$

and from 12)

$$28) \quad 2 \frac{dy}{d\alpha} = (y + c + 1)^2$$

or, if  $y + c + 1 \neq 0$

$$29) \quad y + c + 1 = \frac{2}{\alpha_0 - \alpha}$$

so that

$$30) \quad \beta + (c+1)(\alpha - \alpha_0) = -2 \ln(\alpha_0 - \alpha) + \beta_0$$

From 10)

$$31) \quad \frac{\alpha'}{(\alpha_0 - \alpha)^2} = -\frac{\beta}{\lambda^2} \quad ; \quad \beta \equiv \frac{A e^{-\beta_0 - \alpha_0 (c+1)}}{\phi_0 c}$$

$$\frac{1}{\alpha_0 - \alpha} = \frac{\beta}{\lambda} + D$$

4) requires

$$32) \quad \frac{\alpha'}{\alpha - \alpha_0} - \frac{1}{\lambda} = 0$$

hence  $D = 0$ .

The solution is thus

$$I. \left\{ \begin{array}{l} 33) \quad \alpha = \alpha_0 - \frac{\Delta}{\beta} \\ 34) \quad \beta = \beta_0 - 2 \ln \frac{\Delta}{\beta} + (c+1) \frac{\rho}{\beta} \\ 35) \quad \phi = \phi_0 e^{c\alpha_0 - c\frac{\Delta}{\beta}} \\ \quad \quad c = \frac{1 \pm \sqrt{2\omega - 3}}{\omega - 2} \end{array} \right.$$

It should be noted that reality conditions require  $\omega \geq \frac{3}{2}$  for this solution since  $\omega < \frac{3}{2}$ ,  $\phi \sim e^{c\alpha}$  makes either  $\phi$  or  $\alpha$  complex.

#### E. Spatial Inversion and Type Four Solution

As in the Einstein case, the transformation  $\lambda \rightarrow \bar{\lambda} = \frac{q}{\lambda}$

with

$$36) \quad \begin{aligned} \phi(\lambda) &\rightarrow \bar{\phi}(\bar{\lambda}) = \phi(\lambda) \\ e^{\alpha(\lambda)} &\rightarrow e^{\bar{\alpha}(\bar{\lambda})} = e^{\alpha(\lambda)} \\ e^{\beta(\lambda)} &\rightarrow e^{\bar{\beta}(\bar{\lambda})} = \frac{\Delta}{\lambda} e^{\beta(\lambda)} \end{aligned}$$

leaves the form of the metric unchanged for  $\lambda^2 \neq 0$ . For  $\lambda^2 = 0$ ,  
new metric is

$$\text{IV. } \left\{ \begin{array}{l} 37) \quad \alpha = \alpha_0 - \frac{1}{B\lambda} \\ 38) \quad \beta = \beta_0 + \frac{c+1}{B\lambda} \\ 39) \quad \phi = \phi_0 e^{\alpha_0 c - \frac{c}{B\lambda}} \\ c = \frac{1 \pm \sqrt{2\omega - 3}}{\omega - 2} \end{array} \right.$$

This is just the case,  $y+c+1=0$ , excluded in the derivation of 33), 34), and 35) above.

#### F. Summary

For convenience the four branches of the solution will be tabulated below.  $\alpha_0, \beta_0, \phi_0, C$  and  $B$  are real constants having values in the indicated regions.

$$ds^2 = -e^{\alpha} dt^2 + e^{\beta} (dr^2 + r^2 d\Omega^2)$$

$$\text{I } \left\{ \begin{array}{l} e^{\alpha} = e^{\alpha_0} \left[ \frac{1 - \frac{2B}{\lambda}}{1 + \frac{2B}{\lambda}} \right]^{\frac{1}{\lambda}} \\ e^{\beta} = e^{\beta_0} \left( 1 + \frac{2B}{\lambda} \right)^2 \left[ \frac{1 - \frac{2B}{\lambda}}{1 + \frac{2B}{\lambda}} \right]^{\frac{2-c-1}{\lambda}} \\ \phi = \phi_0 e^{\alpha_0 c} \left[ \frac{1 - \frac{2B}{\lambda}}{1 + \frac{2B}{\lambda}} \right]^{\frac{c}{\lambda}} \\ \lambda^2 \equiv (c+1)^2 - c \left( 1 + \frac{\omega c}{2} \right) > 0 \end{array} \right.$$

$$\text{II} \left\{ \begin{array}{l} \alpha = \alpha_0 + \frac{2}{\Lambda} \tan^{-1} \frac{\Lambda}{2B} \\ \beta = \beta_0 - \frac{2(c+1)}{\Lambda} \tan^{-1} \frac{\Lambda}{2B} - \ln \frac{\Lambda^2}{\Lambda^2 + 4B^2} \\ \phi = \phi_0 e^{\alpha_0 c} e^{\frac{2c}{\Lambda} \tan^{-1} \frac{\Lambda}{2B}} \\ \Lambda^2 \equiv c \left(1 + \frac{\omega c}{2}\right) - (c+1)^2 > 0 \end{array} \right.$$

$$\text{III} \left\{ \begin{array}{l} \alpha = \alpha_0 - \frac{\Lambda}{B} \\ \beta = \beta_0 - 2 \ln \frac{\Lambda}{B} + (c+1) \frac{\Lambda}{B} \\ \phi = \phi_0 e^{c\alpha_0 - c \frac{\Lambda}{B}} \\ c = \frac{1 \pm \sqrt{2\omega - 3}}{\omega - 2} \end{array} \right.$$

$$\text{IV} \left\{ \begin{array}{l} \alpha = \alpha_0 - \frac{1}{B\Lambda} \\ \beta = \beta_0 + \frac{c+1}{B\Lambda} \\ \phi = \phi_0 e^{c\alpha_0 - \frac{c}{B\Lambda}} \\ c = \frac{1 \pm \sqrt{2\omega - 3}}{\omega - 2} \end{array} \right.$$

### XIII. Boundary Conditions

#### A. Introduction: Need for Boundary Conditions

In any attempt to interpret a vacuum solution in terms of a real physical situation, it is necessary to evaluate any constants appearing in the general vacuum solution in terms of observable parameters. For example, the constants in the type I isotropic solution were approximately obtained in XII-C, pages 82 f. by comparison with the weak field solution. This latter contained numbers such as  $m$ , referring to integrals over matter tensor components. Further,  $m$  itself was related to an observed inertial mass in IV-E, pages 21 f., IX-C, pages 71 f. There was one other number, however, which was not so evaluated, namely,  $K_0$ . This was found to be the asymptotic value of the locally measured gravitational "constant," or  $\frac{\partial\psi-4}{\partial\psi-3}$  times the asymptotic value of  $\phi$ .

In general relativity, this freedom for  $K_0$  is unimportant since  $K_0$  must be measured only once. However, the main purpose of introducing  $\phi$  was to determine the locally measured gravitational "constant" as completely as possible in terms of the structure of the universe.

This problem was briefly discussed in VI-H, pages 44 f. There it was pointed out that a possible way of overcoming this difficulty might be the imposition of boundary conditions on  $\phi$ .

#### B. Electrostatic Analogue

From Mach's Principle it might be expected that inertial reactions of a particle are in some way related to the presence and propinquity of other matter in the universe (I-A, page 1). In fact, these reactions might decrease if the effective  $\sum \frac{m}{r}$  from the

universe were to decrease. Hence, in a mass shell universe, inertial reactions in regions of space outside of the shell and distant from it might be small. In terms of a local gravitational constant, this might show up in a large  $K$ . Hence for such a universe

$$1) \quad K \rightarrow \infty \quad \text{as } \Lambda \rightarrow \infty$$

or equivalently

$$2) \quad \phi \rightarrow 0 \quad \text{as } \Lambda \rightarrow \infty$$

Further, such a condition at first glance seems to provide  $\phi \sim \frac{M}{R}$  (I-B, pages 1 f.) inside the shell. That this results from  $\phi \rightarrow 0$  as  $\Lambda \rightarrow 0$  is suggested by the electrostatic analogue. Namely, given an equation of the form

$$3) \quad \nabla^2 \phi = -\rho$$

in flat space, the most general solution (regular at the origin) for the spherically symmetric case of an infinitesimally thin shell of total charge  $Q$  and radius  $R$  is

$$4) \quad \phi = \begin{cases} \phi_0 - \frac{Q}{4\pi R} + \frac{Q}{4\pi r} & \text{for } r > R \\ \phi_0 & \text{for } r < R \end{cases}$$

with  $\phi_0$  a constant not determined by the mathematics of the problem. In electrostatics, the value of  $\phi_0$  is physically unimportant but in the case of the scalar field representing the reciprocal of the gravitational constant, this value is directly measurable.

Hence, it is necessary to add some restriction to the mathematics. The natural method of doing this would be the addition of a boundary condition. The choice  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$  seems to be the least arbitrary, but more important, it yields a value of  $\phi_0 = \frac{Q}{4\pi R}$  which would give, for the case of mass and the gravitational constant

$$5) \quad \frac{KM}{R} \sim 1$$

a relation apparently satisfied by the present universe.

It is very important to notice, however, that there are really two types of boundary conditions  $\phi \rightarrow 0$ . First, that  $\phi \rightarrow 0$  somewhere outside of all matter and second, that  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$ . Notice that the second completely determines  $\phi_0$  to be  $\frac{Q}{4\pi R}$  while the first only restricts  $\phi_0$  to a range,

$$6) \quad 0 \leq \phi_0 \leq \frac{Q}{4\pi R}$$

In other words, if  $\phi \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$ , then  $\phi_0$  depends on  $\lambda_0$  as well as M and R

$$7) \quad \phi_0 = \frac{Q}{4\pi} \left( \frac{1}{R} - \frac{1}{\lambda_0} \right)$$

### C. Weak Field General Relativistic Case

An attempt to carry these arguments over to the general relativistic case leads to some difficulties. The space is curved and the metric and scalar field will interact and influence each other. The fact that the above considerations cannot be directly carried over is suggested by the following order of

magnitude argument for approximate solution to the equations

$$8) \quad \phi(R_{\alpha}^{\beta} - \frac{1}{2} \delta_{\alpha}^{\beta} R) = \pi_{\alpha}^{\beta} + \phi'_{\alpha}{}^{\beta}$$

$$9) \quad \square \phi = -\frac{\pi}{2\omega - 3}$$

Let  $\pi_{\alpha}^{\beta}$  represent a spherically symmetric static shell of radius  $R$  and mass  $M$ . In order for the above discussion to be relevant it seems reasonable to require that the lowest order metric be Minkowskian so that 9) becomes

$$10) \quad \nabla^2 \phi = 0$$

outside the shell. If  $K$  is the value of  $\phi$  at the shell, the next order metric will vary near the shell like  $\frac{KM}{\lambda}$ . Hence for the approximation procedure to be valid

$$11) \quad \frac{KM}{R} \ll 1$$

On the other hand,  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$  requires

$$12) \quad \phi(R) \sim \frac{M}{R}$$

contradicting 11).

D. Description of the Model

And Solutions Consistent with Boundary Conditions

The model universe considered corresponds to a static spherically symmetric mass shell between  $R_1$  and  $R_2$  together with relatively small masses near the origin.

Specifically assume that as  $r$  approaches  $R_1$ , from below, the metric components (in isotropic coordinates) and  $\phi$  satisfy

$$e^{2\alpha} \approx 1 - \frac{K_0 m}{4\pi\lambda} ; e^{2\beta} \approx 1 + \frac{\omega-1}{\omega-2} \frac{K_0 m}{4\pi\lambda}$$

$$13) \quad \phi \approx \frac{1}{K} \left( 1 - \frac{K_0 m}{8\pi\lambda(\omega-2)} \right)$$

$$K > 0 ; m > 0$$

$$0 < \frac{K_0 m}{R_1} \ll 1$$

It should be emphasized that 13) will be used only to determine the signature of the field quantities and their derivatives at  $R_1$ .

Further assume that there is a diagonal matter tensor  $T_{\alpha}^{\beta}$  vanishing outside  $R_1 < r < R_2$  and satisfying

$$14) \quad T_{\alpha\beta} \geq 0 ; T_{\alpha}^{\alpha} \leq 0$$

Further, in order to avoid large discrepancies in the deflection of light and perihelion rotation experiments, assume  $|\omega| \gg 1$  so that, for example,

$$15) \quad \frac{\omega-3}{\omega-2} > 0 ; \frac{\omega-2}{\omega-1} > 0$$

Hence at  $R_1$

$$\phi > 0 \quad ; \quad \alpha' > 0 \quad ; \quad \beta' < 0$$

16)

$$(\omega - 3)(\alpha' + \beta') < 0$$

$$(\omega - 3)\dot{\phi} > 0$$

Which of the solutions listed in XII-F might be used for  $\lambda > R_2$ ? It is immediately clear that  $\phi \rightarrow 0$  outside the shell eliminates type II, i. e.,  $\lambda^2 < 0$ . In fact, for type II  $\phi$  varies as the exponential of a  $\tan^{-1}$  of  $r$  so that for all  $r$ ,  $\phi$  can only vary by a non-zero factor. Further, since  $e^{-\beta}$  is bounded, an arbitrarily large proper distance can be attained for sufficiently large  $r$ . In other words, proper distance goes to infinity with  $r$ .

Similar remarks apply to type IV solution. Hence,  $\phi \rightarrow 0$  outside the shell demands type I or III. This then requires a range of  $C$  and  $\omega$  such that  $(C+1)^2 \geq C + \frac{\omega C^2}{2}$

Further  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$  eliminates type I and thus specifies  $C$  to be one of the roots

$$17) \quad C = \frac{1 \pm \sqrt{2\omega - 3}}{\omega - 2}$$

These roots (and thus the solution) can be real only if  $\omega \geq \frac{3}{2}$ .

This, however, contradicts positive contribution of local matter to  $\phi$ , (I-B, pages 1 f.; VI-B, pages 36 f.; X-B, pages 75 f). Nevertheless, this case will be further studied below (XIII-E), and found to contradict the requirements 14).

The other choice,  $\phi \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$  permits the use of type one solutions. This means that the determination of  $\phi$  inside the shell

would depend on  $\rho_0$  as well as the shell parameters. Actually, this solution is also found to violate 14).

### E. Elimination of $2\omega - 3 > 0$

#### And Thus Type III Solution

From  $\square\phi = \frac{-T}{2\omega-3}$  it follows that, for  $r > R_1$ ,

$$18) \quad e^{-2\beta} \dot{\phi} \sqrt{-g} = \frac{1}{2\omega-3} \int_{R_1}^{\lambda} \sqrt{-g} (-T) d\lambda + e^{-2\beta(R_1)} \dot{\phi}(R_1) \sqrt{-g}(R_1)$$

Hence as long as no singularities in the metric occur

$$19) \quad (2\omega-3) \dot{\phi}(\lambda) \geq (2\omega-3) \dot{\phi}(R_1) > 0$$

so that for  $2\omega - 3 > 0$ ,

$$20) \quad \phi > \frac{1}{K}; \quad \dot{\phi} > 0 \quad \text{for } \lambda > R_1$$

In particular, the choice of type III (since it requires  $2\omega - 3 > 0$  and has a monotone  $\phi$ ) for the external solution is incompatible with the boundary condition  $\phi \rightarrow 0$ , provided no singularities or zeros of  $\phi$  occur for  $R_1 < r < R_2$ .

### F. Type I Solution

Consider then  $\omega < -2$ ,  $\lambda^2 > 0$

$$21) \quad \dot{\phi}(\lambda) \leq \dot{\phi}(R_1) < 0$$

Adding XII-4) and XII-5, page 79) and setting  $\dot{z} \equiv \dot{\alpha} + \dot{\beta}$ , yields

$$22) \quad \dot{z} + z^2 + \frac{3z}{\lambda} + \frac{z\phi}{\phi} = \frac{e^{2\beta}}{\phi} (\pi'_1 + \pi'_2) + \frac{\pi e^{2\beta}}{2\omega-3} - \frac{\phi'}{\lambda\phi}$$

It is easily seen that the right hand side of this equation is non-negative. Thus, at every zero of  $z$ ,  $\dot{z} \geq 0$  so that since  $z \geq 0$  at  $R_1$ ,  $z \geq 0$  for  $\lambda > R_1$ , provided no singularities or zeros of  $\phi$  occur between  $R_1$  and  $r$ , and  $r$  remains in the region where  $z$  is single valued and  $C^1$ . Similarly, setting  $\chi \equiv \beta$  in XII-3 page 79 yields

$$23) \quad \dot{\chi} + \frac{\partial \chi}{\lambda} + \frac{\chi^2}{2} + \frac{\chi\phi}{2\phi} = \frac{e^{2\beta}}{\partial\phi} \left[ \pi'_0 + \frac{\pi}{\partial\omega-3} + e^{-2\beta} \left( \phi'z + \frac{\omega\phi'^2}{2\phi} \right) \right]$$

This time the right hand side is non-positive (since  $\omega < -2$ ;  $\pi \leq 0$ ),  $\dot{\chi} \leq 0$  at every zero of  $\chi$ ,  $\chi \leq 0$  at  $R_1$ , so that  $\chi \leq 0$  for  $\lambda > R_1$ , subject to continuity restrictions above.

In summary, for a reasonable distribution between  $R_1$  and  $R_2$ , the initial conditions at  $R_1$  on the external solutions will require

$$\phi \leq 0 \quad ; \quad (\alpha' + \beta') \geq 0 \quad ; \quad \alpha' \geq 0$$

24)

$$\phi > 0 \quad ; \quad \beta' \leq 0$$

at  $R_2$ . From the form  $\phi \sim \left[ \frac{\lambda - 2B}{\lambda + 2B} \right]^{C_2}$ , it is easily seen that  $\phi \rightarrow 0$  outside the shell requires

$$25) \quad R_2 < |2B| \quad ; \quad \frac{B\epsilon}{\lambda} > 0$$

Further,

$$26) \quad \frac{\dot{\phi}}{\phi} \leq 0 \quad ; \quad \dot{\alpha} \geq 0 \quad ; \quad \frac{\dot{\phi}}{\phi} = c \dot{\alpha}$$

imply  $c \leq 0$

so that  $\frac{B}{\lambda} < 0$ . Under these restrictions,  $\dot{\alpha} \geq 0$  and  $\dot{\phi} \leq 0$  are both fulfilled at  $R_2$ . On the other hand  $\dot{\alpha} + \dot{\beta} \geq 0$  requires

$$27) \quad \frac{\lambda - c}{\lambda} \left( \frac{4B}{R_2^2 - 4B^2} \right) \geq \frac{4B}{R_2(R_2 + 2B)}$$

Hence if  $B > 0$ ,  $\lambda < 0$  and

$$28) \quad \left( 1 - \frac{c}{\lambda} \right) \left( \frac{1}{R_2 - 2B} \right) \geq \frac{1}{R_2} > 0$$

so that

$$29) \quad 1 - \frac{c}{\lambda} \leq 0 \quad ; \quad c^2 \geq \lambda^2$$

On the other hand, from the definition of  $\lambda$ , this reduces to

$$30) \quad c^2 \geq c^2 + 2c + 1 - c - \frac{\omega c^2}{2}$$

or

$$31) \quad 0 \geq c + 1 - \frac{c^2 \omega}{2}$$

Such a range for  $c$  will exist only if  $1 + 2\omega \geq 0$ , which is inconsistent with  $\omega < -2$ .

On the other hand,  $B < 0$ ,  $\lambda > 0$ , 27) yields

$$32) \quad \frac{\lambda - c}{\lambda} \geq \frac{R_2 - 2B}{R_2}$$

or

$$33) \quad \frac{c}{\lambda} \leq \frac{2B}{R_2} < -1$$

which again is inconsistent with  $\omega < -2$ .

### G. Summary

In summary, then, for a spherically symmetric mass distribution between  $R_1$  and  $R_2$  the following assumptions:

- 1)  $|\omega| > 2$
  - 2) as  $r$  approaches  $R_1$  from the left,  $\phi$  and the fields approach the weak field solutions 13) corresponding to a positive point mass at rest at the origin<sup>56</sup>,
  - 3) no singularities in any of the field quantities occur between  $R_1$  and  $R_2$  and in this region  $\tau_0 \leq 0$ ,  $\tau \leq 0$ ,
- imply that there exists a constant  $L > 0$  such that  $\phi > L$  for all  $\lambda > R_2$ .

### H. Approximate Evaluation of Local $K$

From Approximate Interior Solution  
and Boundary Condition  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$

Thus the discussion in XIII-G indicates that  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$  will require pressures as big as densities. However, it might be useful to have more quantitative consequences of this boundary condition. To this end, consider a static solid spherical ball and adjust the

<sup>56</sup> All that is really needed is that the fields and their derivatives have signatures corresponding to the weak field case, i. e., that they satisfy 16).

pressure and density terms to make the interior metric the known solution of the Einstein problem. Specifically, assume that throughout the ball,  $\phi \neq 0$ . Hence, dividing XII-3)4)5) page 79, by  $\phi$ , call the resulting right hand side

$$34) \quad K \overline{T}_\alpha^\beta = K \begin{pmatrix} -\overline{\rho} & 0 \\ 0 & \overline{P}_1 \end{pmatrix}$$

with K reciprocal of  $\phi$  at the origin. Hence, if the true mass tensor is  $T_\alpha^\beta = \begin{pmatrix} -\rho & 0 \\ 0 & P_1 \end{pmatrix}$ , then  $P_3 = P_2$  and  $\rho$ ,  $P_1$ ,  $P_2$ , and  $\phi$  are determined by

$$35) \quad K\overline{\rho} = \frac{1}{\phi} \left( \rho + e^{-2\beta} \square \phi - e^{-2\beta} \left( \dot{\phi}' \alpha + \frac{\omega \phi'^2}{\partial \phi} \right) \right)$$

$$36) \quad K\overline{P}_1 = \frac{1}{\phi} \left( P_1 - e^{-2\beta} \square \phi + e^{-2\beta} \left( \dot{\phi}' \alpha - \frac{\omega \phi'^2}{\partial \phi} \right) \right)$$

$$37) \quad K\overline{P}_2 = \frac{1}{\phi} \left( P_2 - e^{-2\beta} \square \phi + e^{-2\beta} \left( \beta' \dot{\phi} + \frac{\dot{\phi}}{\lambda} + \frac{\omega \phi'^2}{\partial \phi} \right) \right)$$

$$38) \quad \square \phi = \frac{\rho - P_1 - 2P_2}{2\omega - 3}$$

Assuming  $\overline{\rho} = \text{constant}$ , the metric is then (setting  $\alpha = \beta = 0$  at the origin)

$$39) \quad e^{2\alpha} = \left[ \frac{3}{5} C_1 + \left( 1 - \frac{3}{5} C_1 \right) \left( \frac{1 - \frac{K\overline{\rho}\Lambda^2}{12}}{1 + \frac{K\overline{\rho}\Lambda^2}{12}} \right) \right]^2$$

$$40) \quad e^{2\beta} = \left( 1 + \frac{K\overline{\rho}\Lambda^2}{12} \right)^2$$

and

$$41) \quad \overline{P}_1 = \overline{\rho} (C_1 e^{-\alpha} - 1)$$

Thus, setting  $\dot{\phi} = 0$  at the origin, 38) gives

$$42) \quad \lambda^2 \dot{\phi} e^{\alpha+\beta} = \int_0^{\lambda} \frac{\rho - p_1 - 2p_2}{2\omega - 3} e^{\alpha+3\beta} \lambda^2 d\lambda$$

Hence

$$43) \quad \dot{\phi} = O\left(\frac{1}{\omega}\right); \quad \rho = \bar{\rho} + O\left(\frac{1}{\omega}\right); \quad p_1 = \bar{p} + O\left(\frac{1}{\omega}\right) = p_2 + O\left(\frac{1}{\omega}\right)$$

where  $f(x) = O(x)$  means  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$

Fitting the derivative of the metric to that of XII-33), XII-34), page 84, at  $r = R$  then yields

$$44) \quad -\frac{1}{B} = \frac{-K\bar{\rho}R(2-3C)}{6\left(1 + \frac{K\bar{\rho}R^2}{12}\right)\left[2-3C_1 + (3C_1-1)\left(1 + \frac{K\bar{\rho}R^2}{12}\right)\right]}$$

$$45) \quad -\frac{2}{R} + \frac{C+1}{B} = -\frac{K\bar{\rho}R}{6\left(1 + \frac{K\bar{\rho}R^2}{12}\right)}$$

It should be recalled that  $C$  is not a free constant but is determined by  $\lambda^2 = 0$  to be  $C = \frac{1 \pm \sqrt{2\omega-3}}{\omega-3}$ . Hence 44) and 45) determine both  $C$  and  $B$  in terms of  $\bar{\rho}$ ,  $R$ , and  $K$ . The remaining continuity condition, on  $\dot{\phi}$ , will then determine  $K$ . To get an idea of what it would be to lowest order in  $\frac{1}{\omega}$ , assume  $\bar{\rho}$ ,  $\bar{p}$  are small.

Then inside

$$46) \quad \phi \approx \frac{1}{K} + \frac{(\rho - 3p)\lambda^2}{6(2\omega - 3)}$$

and at  $R$ ,

$$47) \quad -\frac{C}{B} = \frac{\dot{\phi}}{\phi} \approx \frac{K(\rho - 3P)R}{3(2\omega - 3)}$$

From 45)

$$48) \quad \frac{C+1}{B} = \frac{2}{R} - \frac{K\bar{\rho}R}{6\left(1 + \frac{K\bar{\rho}R^2}{12}\right)} = \frac{2}{R\left(1 + \frac{K\bar{\rho}R^2}{12}\right)}$$

so that

$$49) \quad \frac{K(\rho - 3P)R^2}{3(2\omega - 3)} \approx -\frac{C}{C+1} \frac{2}{\left(1 + \frac{K\bar{\rho}R^2}{12}\right)}$$

where

$$50) \quad C = \frac{1 \pm \sqrt{2\omega - 3}}{\omega - 2}$$

Hence  $\phi \rightarrow 0$  as  $\lambda \rightarrow \infty$  seems to require a relation of the form 49).

This bears some resemblance to the conjecture

$$51) \quad \frac{KM}{R} \sim 1$$

However, in 49) both sides are negative so that  $P > \frac{1}{3}\rho$ .

Hence this model of a universe does not seem too satisfactory and more study will be required. Of course, a more reasonable boundary condition would be  $\phi \rightarrow$  a cosmological solution. Some information about the cosmology will be obtained in the next section.

#### XIV. Cosmology

##### A. Introduction: Dirac's Conjecture

As pointed out in VI-A, pages 31 f., certain numbers associated with the universe, the Eddington numbers, when expressed in "atomic units" ( $\hbar = c = m_e = 1$ ) group according to order of magnitude.

In particular

$$\begin{aligned}
 R_H &\approx 10^{40} \\
 1) \quad \frac{1}{K} &\approx 10^{40} \\
 T &\approx 10^{40} \\
 M_H &\approx 10^{80}
 \end{aligned}$$

where  $R_H$  is the Hubble radius (defined as the reciprocal of the Hubble constant),  $K$  is the gravitational constant,  $T$  is the age of the universe and  $M_H \equiv \frac{4\pi\rho R_H^3}{3}$ , with  $\rho$  the cosmological mass density. From these "coincidences" Dirac suggested that a comprehensive theory might yield approximate relations of the sort

$$2) \quad R_H \approx t \quad ; \quad \frac{1}{K} \approx t \quad ; \quad M_H \approx t^2$$

If the quantity  $\frac{1}{K} \sim \phi$  is assumed to be determined by the mass distribution in the universe in accordance with a relation of the form  $\phi \sim \frac{M}{\lambda}$  for a point mass, then as a rough estimate, it might be expected that  $\phi \sim \frac{M_V}{R_V}$  where  $M_V$  is the mass visible (i.e., causally related) at the origin and  $R_V$  is some maximum visible "distance." This is consistent with Dirac's conjectures if  $R_V \sim R_H$ . In the following, only  $R_H$  will be considered.

Further, defining

$$3) \quad D_V \equiv \frac{K M_V}{R_H} \quad ; \quad D_H \equiv \frac{K M_H}{R_H}$$

it is easily seen that 1) or 2) would give

$$4) \quad D_V \sim D_H \sim t^0 = \text{constant}$$

B. Status of Dirac's Conjecture in Friedmann Universe

Associated with Einstein Equations

Consider in the Einstein case the Friedmann universe

$$5) \quad ds^2 = -dt^2 + \frac{(R(t))^2}{(1 + \frac{\epsilon \Delta^2}{4})^2} dl^2$$

$$dl^2 = \text{flat 3-metric, } \epsilon = \pm 1, 0$$

$$6) \quad T_{\alpha}^{\beta}{}_{;\beta} = 0 ; P=0 \Rightarrow \rho R^3 = M = \text{constant}$$

and the field equations reduce to

$$7) \quad 3(\dot{R}^2 + \epsilon) = \frac{KM}{R}$$

$$\left( \dot{\quad} \equiv \frac{d\quad}{dt} \right)$$

the solutions to which are, neglecting an additive constant for time,

$$8) \quad t = \mu \left[ \sin^{-1} \left( \frac{R}{\mu} \right)^{\frac{1}{2}} - \left( \frac{R}{\mu} \right)^{\frac{1}{2}} \left( 1 - \frac{R}{\mu} \right)^{\frac{1}{2}} \right] ; \epsilon = +1$$

$$9) \quad t = \frac{2}{3} \mu \left( \frac{R}{\mu} \right)^{\frac{3}{2}} ; \epsilon = 0$$

$$10) \quad t = \mu \left[ -\sinh^{-1} \left( \frac{R}{\mu} \right)^{\frac{1}{2}} + \left( \frac{R}{\mu} \right)^{\frac{1}{2}} \left( 1 + \frac{R}{\mu} \right)^{\frac{1}{2}} \right] ; \epsilon = -1$$

with  $\mu \equiv \frac{KM}{3}$ . Considering a test particle near the origin, the observed gravitational constant is easily seen to be exactly  $K$ , which is constant, thus violating Dirac's  $K \sim \frac{1}{t}$  conjecture. On the other hand, for a light ray received at the origin at  $t_0$ , the coordinate  $r_i$  at time of emission  $t_i$  must be related by

$$11) \quad \int_{r_i}^0 \frac{dr}{(1 + \frac{\epsilon r^2}{4})} = - \int_{t_i}^{t_0} \frac{dt}{R(t)}$$

The choice  $t_i = 0$  or  $R(t_i) = 0$  for the earliest emitted light ray then leads to the following relation in the three cases (using  $\dot{R} = \sqrt{\mu - \epsilon R}$ )

$$12) \quad \frac{r_i}{2} = \sqrt{\frac{R(t_0)}{\mu}} \quad ; \epsilon = 0$$

$$13) \quad \tan^{-1} \frac{r_i}{2} = \sin^{-1} \left( \frac{R(t_0)}{\mu} \right)^{\frac{1}{2}} \quad ; \epsilon = +1$$

$$14) \quad \tanh^{-1} \frac{r_i}{2} = \sinh^{-1} \left( \frac{R(t_0)}{\mu} \right)^{\frac{1}{2}} \quad ; \epsilon = -1$$

Hence for the mass in the visible universe,

$$15) \quad M_V(t_0) = 4\pi \int_0^{r_i(t_0)} \frac{\rho R^3 dr^2}{(1 + \frac{\epsilon r^2}{4})^3}$$

$$16) \quad M_V(t_0) = \frac{48\pi t_0}{K} \quad ; \epsilon = 0$$

$$17) \quad M_V(t_0) = \frac{12\pi}{K} \left\{ \sin^{-1} \left( \frac{R(t_0)}{\mu} \right)^{\frac{1}{2}} - \frac{r_i(t_0)}{2} \frac{[1 - (\frac{r_i(t_0)}{2})^2]}{[1 + (\frac{r_i(t_0)}{2})^2]^2} \right\} ; \epsilon = +1$$

$$18) \quad M_V(t_0) = \frac{12\pi}{K} \left\{ -\sinh^{-1} \left( \frac{R(t_0)}{\mu} \right)^{\frac{1}{2}} + \frac{\lambda_i(t_0)}{\alpha} \frac{[1 + (\frac{\lambda_i(t_0)}{\alpha})^2]}{[1 - (\frac{\lambda_i(t_0)}{\alpha})^2]^2} \right\} ; \epsilon = -1$$

For the case  $\epsilon = 0$ ,

$$19) \quad R_H = \frac{3}{\alpha} t ; M_V = \frac{48\pi t}{K} ; M_H = \frac{6\pi t}{K}$$

$$D_H = 4\pi ; D_V = 32\pi$$

Identifying the present value of  $t$  with the age of the universe,  $\sim 10^{40}$ , gives a fit of the observed values of  $R_H$  and  $\rho$ . It should also be noticed that a substitution  $\frac{1}{K} \sim t$  would yield  $M_V \sim M_H \sim t^2$ . For  $\epsilon = +1$  and  $\frac{R}{\mu}$  near 1, the  $\sin^{-1}$  term in 8) is dominant so

$$20) \quad R \sim \mu \sin^2 \frac{t}{\mu} \quad \text{as} \quad \frac{t}{\mu} \rightarrow \frac{\pi}{2}$$

The Hubble radius and mass are now

$$R_H \approx \frac{KM}{6} \tan \frac{t}{\mu}$$

$$21) \quad M_H \approx \frac{4\pi \mu}{3 \left( \sin \frac{2t}{\mu} \right)^3}$$

and both are rapidly increasing functions of time. The dominant contribution to the visible mass in this range is the  $\sin^{-1}$  term so that

$$22) \quad M_V \sim \frac{12\pi t}{K}$$

Hence

$$23) \quad D_H \sim \frac{\pi}{\sin^4\left(\frac{t}{2t_0}\right) \cos^2\left(\frac{t}{2t_0}\right)} \xrightarrow{\frac{t}{2t_0} \rightarrow \frac{\pi}{2}} \infty$$

$$24) \quad D_V \sim \frac{24\pi t}{M \tan \frac{t}{2t_0}} \xrightarrow{\frac{t}{2t_0} \rightarrow \frac{\pi}{2}} 0$$

For  $\epsilon = -1$ ,  $R \rightarrow \infty$ ,  $R/t \rightarrow 1$  so that

$$R_H \rightarrow t$$

$$M_H \rightarrow \frac{4\pi M}{K}$$

$$25) \quad M_V \rightarrow \frac{24\pi t^2}{KM}$$

$$D_H \rightarrow \frac{4\pi M}{t}$$

$$D_V \rightarrow \frac{24\pi t}{M}$$

### C. Friedmann Analogue in $\phi$ Field Theory

For the scalar theory the field equations become

$$26) \quad 3(\dot{R}^2 + \epsilon) = \frac{M}{\phi R} - \frac{3\dot{R}R\dot{\phi}}{\phi} - \frac{\omega R^2}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2$$

$$27) \quad R^3 \ddot{\phi} + 3\dot{R}R^2 \dot{\phi} = -\frac{M}{2\omega - 3}$$

$$28) \quad \rho R^3 \equiv M = \text{constant}$$

From 27)

$$29) \quad \dot{\phi} R^3 = - \frac{M(t-t_0)}{2\omega-3} \quad ; t_0 = \text{constant}$$

If this were the only restriction on  $\phi$  and  $R$ , all the requirements of Dirac's cosmology could be satisfied for  $\epsilon = 0$ , since  $\phi \sim t$ ,  $R \sim t^{\frac{1}{3}}$ ,  $R_H \sim t$ ,  $M_H \sim t^2$ ,  $M_V \sim t^2$  are compatible with it.

Appropriate initial value data would consist of the values of  $\phi$ ,  $\dot{\phi}$ ,  $\rho$ , and  $R$  at some time. Hence the general solution must contain three independent constants, apart from  $M$ . Further, if  $\phi(t)$ ,  $\rho(t)$ ,  $R(t)$  is any solution, so also is  $C_1 \phi(C_2 t)$ ,  $C_1 C_2^2 \rho(C_2 t)$ ,  $\frac{R(C_2 t)}{C_2}$  for any constants  $C_1, C_2$ . Hence, the field equations above, as would be expected, cannot yield any unique values for  $\phi$ ,  $\rho$ ,  $R$  but only relations involving their ratios.

Although these equations are reducible to one first order ordinary differential equation, its form, (with  $y \equiv \dot{R}$ ,  $X \equiv \frac{R}{t-t_0}$ ,  $\dot{y} \equiv \frac{dy}{dx}$ )

$$30) \quad 3(y^2 + \epsilon) = \left[ \frac{\dot{y}x(y-x) + 2y^2 + 2\epsilon}{\omega-1 + \frac{y}{x}} \right] \left[ 2\omega-3 + \frac{3y}{x} - \frac{\omega}{2x^2} \left( \frac{\dot{y}x(y-x) + 2y^2 + 2\epsilon}{\omega-1 + \frac{y}{x}} \right) \right]$$

seems to preclude exact general solution in terms of elementary functions.

#### D. Special Solutions

There is an exact special solution, however, (only one free constant apart from  $M$  and  $t_0$ ) for  $\epsilon = 0$ , namely,

$$31) \quad \phi = C_0 (t-t_0)^{\frac{1}{3}}$$

$$32) \quad R = - \left( \frac{M}{(2\omega-3)\pi C_0} \right)^{\frac{1}{3}} (t-t_0)^{\frac{2-\omega}{3}}$$

with either

$$33) \quad n = -4$$

or

$$34) \quad n = -\frac{2}{3\omega-4}$$

In both cases,  $R_H \sim (t-t_0)$  so that the present  $t-t_0$  must be  $\sim 10^{40}$ , and  $C_0 \sim 10^{200}$  or  $C_0 \sim 10^{40(1+\frac{2}{3\omega-4})}$ . However, to fit both the perihelion rotation and light deflection results within 10% of the Einstein value  $\omega < -4$  or  $\omega > 8$  so that in all cases  $n < 2$ .

Further, from 28)

$$35) \quad \rho = -m C_0 (t-t_0)^{n-2} (2\omega-3)$$

or

$$36) \quad \frac{\rho}{\phi} = \frac{-m(2\omega-3)}{(t-t_0)^2}$$

so that

$$37) \quad R_H = \frac{3}{2-m} (t-t_0)$$

$$38) \quad M_H = -\frac{4\pi}{3} C_0 n (2\omega-3) \left(\frac{3}{2-m}\right)^3 t^{n+1}$$

and

$$39) \quad D_H = -\frac{4\pi n (2\omega-3)}{3} \left(\frac{3}{2-m}\right)^2$$

The choice  $n = -4$  requires  $2\omega-3 > 0$  for  $\rho > 0$ ,  $\phi > 0$ , while  $|\omega| > \frac{3}{2}$ ,  $n = -\frac{2}{3\omega-4}$ , leaves the sign of  $2\omega-3$  undetermined. The coordinate radius of the visible universe is now infinite for  $n = -4$ , while for

$n = -\frac{2}{3\omega-4}$ , it is (assuming  $\frac{\omega-2}{3\omega-4} > 0$ )

$$40) \quad r_v = \left( \frac{2C_0(2\omega-3)}{(3\omega-4)M} \right)^{\frac{1}{3}} \left( \frac{3\omega-4}{\omega-2} \right) (t-t_0)^{\frac{\omega-2}{3\omega-4}}$$

Thus

$$41) \quad M_v = \frac{8\pi C_0(2\omega-3)(3\omega-4)}{3(\omega-2)^2} (t-t_0)^{\frac{3\omega-2}{3\omega-4}}$$

$$42) \quad D_v = \frac{16\pi}{3} \frac{(2\omega-3)(3\omega-4)(\omega-1)}{(\omega-2)^3}$$

There is a corresponding exact solution for  $\epsilon \neq 0$  namely (setting  $t_0 = 0$  for simplicity)

$$43) \quad \phi = C_0 (a+t^2)^{-2}$$

$$R = B (a+t^2)$$

with

$$44) \quad a \equiv \frac{3\epsilon}{(2\omega-3)^{\frac{1}{2}}} \left( \frac{C_0}{2M} \right)^{\frac{2}{3}}$$

and

$$45) \quad B \equiv \left( \frac{M}{4(2\omega-3)C_0} \right)^{\frac{1}{3}}$$

Hence

$$R_H = \frac{a+t^2}{2t}$$

$$46) \quad \rho = \frac{4C_0(2\omega-3)}{(a^2+t^2)^3} ; D_H = \frac{4\pi}{3}(2\omega-3) \left( \frac{a+t^2}{t^2} \right)$$

$$M_H = \frac{2\pi(2\omega-3)C_0}{3t^3}$$

For  $a > 0$ ,  $\epsilon(2\omega-3) > 0$ , while  $\rho > 0$  and  $C_0 > 0$  imply  $(\partial\omega-3) > 0$ . Thus  $a > 0$  requires  $\epsilon > 0$ ,  $\partial\omega-3 > 0$ . Since there are now no zeros of  $R(t)$ , the lower limit of time in the visible radius determination must be  $-\infty$ . Hence

$$47) \quad \tan^{-1} \frac{\lambda_V}{\partial} = \sqrt{\frac{\partial\omega-3}{3}} \left( \tan^{-1} \frac{t}{\sqrt{a}} + \frac{\pi}{\partial} \right)$$

and

$$48) \quad M_V(t) = 4\pi M \left[ \tan^{-1} \frac{\lambda_V}{\partial} - \frac{\lambda_V}{\partial} \frac{(1 - \frac{\lambda_V^2}{4})}{(1 + \frac{\lambda_V^2}{4})^2} \right]$$

The fact that for  $t > 0$ ,  $\lambda_V$  is negative is essentially due to the fact that the coordinate system  $r, \theta, \phi$  is not globally defined. However, this is only a formal difficulty and is discussed in appendix A.

The first term is the dominant term for large  $\omega$  in the right hand side of 48) since the second is bounded by  $\pm \frac{1}{4}$ . Hence

$$49) \quad M_V \simeq 8\pi M \sqrt{\frac{\partial\omega-3}{3}} \left( \frac{\pi}{\partial} + \tan^{-1} \frac{t}{\sqrt{a}} \right)$$

From 46),  $R_H > 0$  requires  $t > 0$ . For  $\frac{t}{\sqrt{a}} \ll 1$ ,

$$M_V \simeq 8\pi M \sqrt{\frac{\partial\omega-3}{3}} \left( \frac{\pi}{\partial} + \frac{t}{\sqrt{a}} \right)$$

$$\phi \simeq \frac{C_0}{a^2} \left( 1 - 2 \left( \frac{t}{\sqrt{a}} \right)^2 \right)$$

$$R_H \simeq \frac{a}{\partial t} \left( 1 + \left( \frac{t}{\sqrt{a}} \right)^2 \right)$$

$$50) \quad M_H = \frac{2\pi(2\omega-3)C_0}{3t^2}$$

$$D_H \simeq \frac{4\pi(2\omega-3)}{3 \left( \frac{t}{\sqrt{a}} \right)^2} \left( 1 + \left( \frac{t}{\sqrt{a}} \right)^2 \right) \gg 1$$

$$D_V \simeq 24\pi \left( \frac{\pi}{\partial} + \frac{t}{\sqrt{a}} \right) \frac{t}{\sqrt{a}} \ll 1$$

To fit present values to lowest order in  $\frac{t}{\sqrt{a}}$  set

$$M = \lambda_1 \pi^2 \quad ; \quad C_0 = \lambda_2 \pi^5$$

$$a = \frac{3}{(2\omega-3)^{\frac{1}{2}}} \left( \frac{\lambda_2}{\partial \lambda_1} \right)^{\frac{2}{3}} \pi^2$$

51)

$$t = \frac{3}{2(2\omega-3)^{\frac{1}{2}}} \left( \frac{\lambda_2}{\partial \lambda_1} \right)^{\frac{2}{3}} \pi$$

so that to lowest order

$$R_H = \pi$$

$$\phi = \pi \left[ \frac{\lambda_2}{9} \left( \frac{\partial \lambda_1}{\lambda_2} \right)^{\frac{4}{3}} (2\omega-3)^{\frac{2}{3}} \right]$$

$$M_H = \frac{16\pi (2\omega-3)^2 \lambda_2}{81} \left( \frac{\partial \lambda_1}{\lambda_2} \right)^2 \pi^2$$

52)

$$D_H = \frac{16\pi (2\omega-3)^{\frac{4}{3}}}{9} \left( \frac{\partial \lambda_1}{\lambda_2} \right)^{\frac{2}{3}}$$

$$M_V = 4\pi^2 \lambda_1 \left( \frac{2\omega-3}{3} \right)^{\frac{1}{2}} \pi^2$$

$$D_V = \frac{6\pi^2 3^{\frac{1}{2}}}{(2\omega-3)^{\frac{1}{2}}} \left( \frac{\lambda_2}{\partial \lambda_1} \right)^{\frac{1}{3}}$$

Hence, while present values can be fit by  $T \sim 10^{40}$ ,  $\lambda_2 \sim \lambda_1^4 \omega^2$ ;  $\lambda_1 \sim 1$  there is no natural finite age for the universe and the detailed Dirac relationships are satisfied only at present. Similarly, for  $\frac{\sqrt{a}}{t} \ll 1$

$$\begin{aligned}
 M_V &\approx 8\pi M \sqrt{\frac{2\omega-3}{3}} \left( \pi - \frac{\sqrt{a}}{t} \right) \\
 R_H &\approx \frac{t}{3} \left( 1 + \left( \frac{\sqrt{a}}{t} \right)^2 \right) \\
 53) \quad \phi &\approx \frac{C_0}{t^4} \left( 1 - 2 \left( \frac{\sqrt{a}}{t} \right)^2 \right) \\
 M_H &\approx \frac{2\pi (2\omega-3) C_0}{3 t^3} \\
 D_H &\approx \frac{4\pi (2\omega-3)}{3} \left( 1 + \left( \frac{\sqrt{a}}{t} \right)^2 \right) \\
 D_V &\approx \frac{16 t^3 M (2\omega-3)^{\frac{1}{2}}}{C_0 3^{\frac{1}{2}}} \left( \pi - \frac{\sqrt{a}}{t} \right) \left( 1 + \left( \frac{\sqrt{a}}{t} \right)^2 \right)
 \end{aligned}$$

Again, only present values can be fit by  $C_0 \approx \pi^5$ ,  $t = \pi$  except that now  $D_H \sim 1$  is approximately satisfied for  $\frac{t}{\sqrt{a}}$  sufficiently large.

For  $a < 0$ ,  $\epsilon = \pm 1$ ,  $2\omega-3 < 0$ , and  $\rho > 0$ ,  $\phi > 0$ ,  $R_H > 0$  imply

$$54) \quad 0 > t > -\sqrt{-a}$$

and, using the notation of appendix A (with  $t_L$  = lower limit of integration)

$$55) \quad \Psi_V(t) = -\sqrt{\frac{-2\omega+3}{3}} \left[ \ln \left( \frac{\sqrt{-a}-t}{\sqrt{-a}+t} \right) - \ln \left( \frac{\sqrt{-a}-t_L}{\sqrt{-a}+t_L} \right) \right]$$

Thus, since

$$56) \quad M_V \xrightarrow[\Psi_V \rightarrow \infty]{} \Psi_V$$

as  $t_L \rightarrow -\sqrt{-a}$

$$57) \quad M_V \rightarrow \infty$$

Similarly for  $a < 0$ ,  $\epsilon = -1$ ,  $2\omega - 3 > 0$

$$t > \sqrt{-a}$$

58)

$$\partial \tanh^{-1} \frac{\lambda v}{\partial} = \sqrt{\frac{2\omega - 3}{3}} \left[ \ln \left( \frac{t - \sqrt{-a}}{t + \sqrt{-a}} \right) - \ln \left( \frac{t_L - \sqrt{-a}}{t_L + \sqrt{-a}} \right) \right]$$

and

$$59) \quad M_V = 4\pi M \left[ \frac{\lambda v}{\partial} \frac{(1 + \frac{\lambda v^2}{4})}{(1 - \frac{\lambda v^2}{4})^2} - \tanh^{-1} \frac{\lambda v}{2} \right]$$

so that as  $t_L \rightarrow \sqrt{-a}$ ,  $M_V \rightarrow \infty$ .

### E. Expansion of General Solution in Powers of $\frac{1}{\omega}$

To estimate general solutions the first few terms in an expansion in powers of  $\frac{1}{\omega}$  will be given.

$$R = \omega^a \left( R_0 + \frac{R_1}{\omega} + \frac{R_2}{\omega^2} + \dots \right)$$

60)

$$\phi = \omega^b \left( \phi_0 + \frac{\phi_1}{\omega} + \frac{\phi_2}{\omega^2} + \dots \right)$$

Before beginning this, however, it is convenient to notice some facts about the constants of integration in an extended type of perturbation procedure since it is important to see whether new constants

are meaningfully introduced at any stage.

In appendix B it is shown that if in a perturbation calculation of the solution of  $n$  simultaneous first order ordinary differential equations the most general solution (which may contain fewer than  $n$  constants) has been obtained to the lowest order equations, then only special solutions in higher order need be used. In other words, the extra constants in higher order solutions are not independent of those contained in the lowest order solution. Although this is trivial for linear equations with all constants additive, it is not obvious in some of the cases considered below.

$$F. \quad \underline{\epsilon = 0; 3a + b = -1; \dot{\phi}_0 \neq 0}$$

29) yields to first order in  $\frac{1}{\omega}$

$$61) \quad \omega^{3a+b} \dot{\phi}_0 R_0^3 = - \frac{M(t-t_0)}{2\omega}$$

Hence, for  $\dot{\phi}_0 \neq 0$

$$62) \quad 3a + b = -1 \quad ; \quad R_0^3 = - \frac{M(t-t_0)}{2\dot{\phi}_0}$$

For  $\epsilon = 0$ , the leading terms on 26) are of order  $\omega$  and reduce to

$$63) \quad - \frac{2\omega \dot{\phi}_0}{(t-t_0)\phi_0} - \frac{\omega}{\partial} \left( \frac{\dot{\phi}_0}{\phi_0} \right)^2 = 0$$

so that either

$$64) \quad \phi_0 = c_0 = \text{constant}$$

or

$$65) \quad \phi_0 = C_0 (t - t_0)^{-4}$$

However, this latter is simply the exact solution 31) with  $n = -4$  found above.

$$G. \quad \underline{\epsilon = 0; a = b = 0; \dot{\phi}_0 = 0}$$

Consider now 64), requiring  $3a + b = 0$ ,

$$66) \quad \dot{\phi}_1 R_0^3 = -\frac{M(t - t_0)}{2}$$

Further,  $\omega^0$  terms in 26) yield

$$67) \quad 3 \left( \frac{\dot{R}_0}{R_0} \right)^2 = \frac{M}{C_0 R_0^3}$$

so that

$$68) \quad R_0 = \left( \frac{3M}{4C_0} \right)^{\frac{1}{3}} (t - t_1)^{\frac{2}{3}}$$

with  $t_1 = \text{constant}$  and

$$69) \quad \phi_1 = -\frac{2}{3} C_0 \left[ \ln(t - t_1) - \frac{t_1 - t_0}{t - t_1} + C_1 \right]$$

26) now becomes to order  $-1$

$$70) \quad 6 \frac{\dot{R}_0}{R_0} \frac{d}{dt} \left( \frac{R_1}{R_0} \right) = -\frac{M}{C_0 R_0^3} \left( \frac{\phi_1}{C_0} + \frac{3R_1}{R_0} \right) - \frac{3\dot{R}_0 \dot{\phi}_1}{R_0 C_0} - \frac{(\dot{\phi}_1)^2}{2C_0^2}$$

so that

$$71) \quad R_1 = R_0 \left[ \frac{2}{9} (\ln(t-t_1) + C_1) + \frac{1}{18} \left( 1 + \left( \frac{t_1-t_0}{t-t_1} \right)^2 \right) + \frac{C_2}{t-t_1} \right]$$

and

$$72) \quad \dot{\phi}_2 + \frac{3R_1}{R_0} \dot{\phi}_1 = \frac{3}{\omega} \dot{\phi}_1$$

From appendix B,  $C_1$  and  $C_2$  can be arbitrarily fixed in 69),  
71). In particular, if  $\frac{t_1-t_0}{t-t_1}$  is not large, in order to make the expansion meaningful at present,  $C_1 = -\ln(\mathcal{T}-t_1)$  where  $\mathcal{T}$  is one of the order of magnitude of the present time and  $C_2$  can be set at zero. Hence, setting  $a = b = 0$  gives

$$73) \quad \phi = C_0 \left\{ 1 - \frac{2}{3\omega} \left[ \ln \left( \frac{t-t_1}{\mathcal{T}-t_1} \right) - \left( \frac{t_1-t_0}{t-t_1} \right) + \frac{2}{9\omega^2} \left( \ln \frac{t-t_1}{\mathcal{T}-t_1} \right)^2 - \left( \ln \left( \frac{t-t_1}{\mathcal{T}-t_1} \right) \right) \left( 4 + \frac{2(t_1-t_0)}{t-t_1} \right) + 2 \left( \frac{t_1-t_0}{t-t_1} \right) - \frac{1}{4} \left( \frac{t_1-t_0}{t-t_1} \right)^2 - \frac{1}{6} \left( \frac{t_1-t_0}{t-t_1} \right)^3 \right] + \dots \right\}$$

$$R = \left( \frac{3M}{4C_0} \right)^{\frac{1}{3}} (t-t_1)^{\frac{2}{3}} \left\{ 1 + \frac{1}{9\omega} \left[ 2 \ln \left( \frac{t-t_1}{\mathcal{T}-t_1} \right) + \frac{1}{3} \left( 1 + \left( \frac{t_1-t_0}{t-t_1} \right)^2 \right) \right] + \dots \right\}$$

$$\rho = \frac{4C_0}{3(t-t_1)^2} \left\{ 1 - \frac{1}{3\omega} \left[ 2 \ln \left( \frac{t-t_1}{\mathcal{T}-t_1} \right) + \frac{1}{3} \left( 1 + \left( \frac{t_1-t_0}{t-t_1} \right)^2 \right) \right] + \dots \right\}$$

$$R_H = \frac{3(t-t_1)}{\omega} \left\{ 1 - \frac{1}{3\omega} \left[ 1 - \frac{1}{3} \left( \frac{t_1-t_0}{t-t_1} \right)^2 \right] + \dots \right\}$$

73 cont'd)

$$M_H = 6\pi(t-t_1)C_0 \left\{ 1 - \frac{1}{3\omega} \left[ 2 \ln \left( \frac{t-t_1}{T-t_1} \right) + \frac{7}{2} - \left( \frac{t_1-t_0}{t-t_1} \right)^2 \right] + \dots \right\}$$

$$D_H = 4\pi \left\{ 1 - \frac{1}{\omega} \left[ \frac{5}{6} + \frac{2}{3} \left( \frac{t_1-t_0}{t-t_1} \right) + \frac{1}{2} \left( \frac{t_1-t_0}{t-t_1} \right)^2 \right] + \dots \right\}$$

This is a general solution since there are four independent constants  $M$ ,  $C_0$ ,  $t_0$ ,  $t_1$  and present values can be fit with  $t-t_1 \sim 10^{40}$ ;  $C_0 \sim 10^{40}$ . For  $t_0 = t_1$ , it is seen to be simply an expansion of the special solution 31) 32) with  $n = -\frac{2}{3\omega-4}$  and with proper adjustment of the constants. For  $t_0 \neq t_1$ , no information can be obtained about the radius and mass of the visible universe since the procedure is clearly divergent for small  $t-t_1$ .

$$H. \quad \underline{\epsilon = \pm 1; a = 0; \dot{\phi}_0 \neq 0}$$

Consider now  $\epsilon = \pm 1$ . The  $\frac{\epsilon}{R}$  term in 26) requires that  $a = \frac{n}{2}$ , with  $n$  an integer or zero. For  $\dot{\phi}_0 \neq 0$ ,  $a = 0$ , it is seen that 26) is a first order differential equation for  $\phi_m$  and involves only  $R_m$  with  $m < n$  while 29) gives  $R_n$  in terms of  $\phi_m$  in a zeroth order equation (i.e., no free constants). Hence this procedure will yield only a special solution. In fact, from 26)

$$74) \quad \phi_0 = C_0 (t-t_0)^{-4}$$

$$75) \quad R_0 = \left( \frac{M}{8C_0} \right)^{\frac{1}{3}} (t-t_0)^2$$

Terms of order  $\omega^0$  in 26) reduce to

$$76) \quad \frac{3\epsilon}{\left(\frac{M}{8C_0}\right)^{\frac{2}{3}}(t-t_0)^2} = \frac{2}{(t-t_0)} \frac{d}{dt} \left(\frac{\phi_1}{\phi_0}\right)$$

so that, neglecting the constant, (see appendix B)

$$77) \quad \phi_1 = -\frac{3\epsilon}{4} C_0 \left(\frac{8C_0}{M}\right)^{\frac{2}{3}} (t-t_0)^{-6}$$

and

$$78) \quad R_1 = \frac{1}{2} R_0 \left(1 + \frac{3\epsilon}{4(t-t_0)^2} \left(\frac{8C_0}{M}\right)^{\frac{2}{3}}\right)$$

However, this is simply an expansion of the special solution 43) with the substitutions

$$79) \quad \begin{aligned} C_0 &\rightarrow C_0 \\ t &\rightarrow t-t_0 \\ M &\rightarrow \frac{4M\omega^3}{(2\omega-1)^2} \\ \text{I. } \underline{\epsilon = \pm 1; a = 0; \dot{\phi}_0 = 0} \end{aligned}$$

For  $a = 0$ ,  $\phi_0 = C_0$ ,  $b = 0$  and to terms of order  $\omega^0$ , 26) reduces to the Einstein equation for  $R_0$  with  $\frac{1}{C_0}$  replacing  $K$  and

$$80) \quad \dot{\phi}_1 = -\frac{M(t-t_0)}{2R_0^3}$$

Hence expressing  $t-t_1$ , as a function of  $R_1$  through 8) or 10),

$$81) \quad \phi_1 = -3C_1 \int \frac{[\sin^{-1}\sigma - \sigma(1-\sigma^2)^{\frac{1}{2}} + (t_1-t_0)\left(\frac{3C_0}{M}\right)] d\sigma}{\sigma^4(1-\sigma^2)^{\frac{1}{2}}}; \quad \epsilon = +1$$

$$82) \quad \phi_1 = -3C_0 \int \frac{[\sigma(1+\sigma^2)^{\frac{1}{2}} - \sinh^{-1} \sigma + (t_1 - t_0) \left(\frac{3C_0}{M}\right)] d\sigma}{\sigma^4 (1+\sigma^2)^{\frac{1}{2}}} \quad ; \epsilon = -1$$

where

$$83) \quad \sigma \equiv \left(\frac{3R_0 C_0}{M}\right)^{\frac{1}{2}}$$

so that for  $\epsilon = +1$

$$84) \quad \frac{\phi_1}{C_0} = \frac{[\sinh^{-1} \sigma - \sigma(1-\sigma^2)^{\frac{1}{2}}] (1-\sigma^2)^{\frac{1}{2}} (1+2\sigma^2)}{\sigma^3} + \\ + 1 - 2\sigma^2 - 2 \ln \sigma + \\ + \frac{9C_0(t_1 - t_0)}{M} \left[ \frac{(1-\sigma^2)^{\frac{1}{2}}}{3\sigma^3} + \frac{2}{3} \frac{(1-\sigma^2)^{\frac{1}{2}}}{\sigma} \right] + C_1$$

and, for  $\epsilon = -1$

$$85) \quad \frac{\phi_1}{C_0} = \frac{[\sigma(1+\sigma^2)^{\frac{1}{2}} - \sinh^{-1} \sigma] (1+\sigma^2)^{\frac{1}{2}} (1-2\sigma^2)}{\sigma^3} + \\ + 1 + 2\sigma^2 - 2 \ln \sigma + \\ + \frac{9C_0(t_1 - t_0)}{M} \left[ \frac{(1+\sigma^2)^{\frac{1}{2}}}{3\sigma^3} + \frac{2}{3} \frac{(1+\sigma^2)^{\frac{1}{2}}}{\sigma} \right] + C_1$$

From 26) it is seen that  $R_1$  can be taken as  $\frac{1}{2} \dot{R}_0$  (since  $\frac{\partial R_0}{\partial t_1} = -\dot{R}_0$ ) with  $f$  given by

$$86) \quad f = -\frac{1}{6} \int \frac{R_0^2}{\dot{R}_0^2} \left[ \frac{M}{C_0 R_0^3} \frac{\phi_1}{C_0} + \frac{3\dot{R}_0}{R_0} \frac{\dot{\phi}_1}{C_0} + \frac{\dot{\phi}_1^2}{2C_0^2} \right] dt$$

Although it is possible that the integration can be performed in terms of elementary functions, its form will undoubtedly be unwieldy. However,

several limiting cases will be considered. For small  $\sigma$ ,  $\sigma^3 \rightarrow \frac{9(t-t_1)C_0}{2M}$  and the leading terms in  $R_0, \phi_1$ , approach the values in 89), 91) below. Further, 86) differs from the corresponding equation determining  $R_1$  in 89) only through the  $\epsilon$  terms, in the latter. Hence as  $t - t_1 \rightarrow 0$ , this case yields relations 94) with  $\epsilon = 0$  to lowest order in  $t - t_1$ .

For  $\epsilon = +1$ ,  $\sigma$  near 1,  $t - t_1 \rightarrow \frac{\pi M}{6C_0}$  and the first few terms in an expansion about  $t = t_1 + \frac{\pi M}{6C_0}$  are

$$\begin{aligned}
 \phi &= C_0 - \frac{27C_0^3}{2M^2\omega} \left\{ (t-t_1 - \frac{\pi M}{6C_0})(t_1-t_0 + \frac{\pi M}{6C_0}) + \right. \\
 &\quad \left. + (t-t_1 - \frac{\pi M}{6C_0})^2 \left[ \frac{1}{\omega} + \frac{27C_0^2}{8M^2} (t_1-t_0 + \frac{\pi M}{6C_0}) \right] + \dots \right\} + \dots \\
 87) \quad R &= \left\{ \frac{M}{3C_0} - \frac{3C_0}{4M} (t-t_1 - \frac{\pi M}{6C_0})^2 + \dots \right\} + \\
 &\quad + \frac{1}{\omega} \left\{ -\frac{3^5 C_0^5}{48 M^3} (t_1-t_0 + \frac{\pi M}{6C_0})^2 (t-t_1 - \frac{\pi M}{6C_0}) + \dots \right\} + \dots
 \end{aligned}$$

For  $\epsilon = -1$ ,  $\sigma$  large,  $\frac{R_0}{t-t_1} \rightarrow 1$ , and the dominant term in 85) is the logarithm.

$$\begin{aligned}
 \phi &\rightarrow C_0 \left( 1 - \frac{1}{\omega} \ln(t-t_1) \right) \\
 R_H &\rightarrow (t-t_1) \left[ 1 + \frac{1}{\omega} \left( \frac{f}{t-t_1} - \frac{1}{6(t-t_1)} \left( \frac{M \ln(t-t_1)}{C_0} + \frac{5}{2(t-t_1)} \right) \right) \right] \\
 88) \quad \rho &\rightarrow \frac{M}{(t-t_1)^3} \left( 1 - \frac{3f}{\omega(t-t_1)} \right) \\
 M_H &\rightarrow \frac{4\pi M}{3} \left[ 1 - \frac{1}{2\omega(t-t_1)} \left( \frac{M \ln(t-t_1)}{C_0} + \frac{5}{2(t-t_1)} \right) \right] \\
 D_H &\rightarrow \frac{4\pi M}{3C_0(t-t_1)} \left[ 1 + \frac{1}{\omega} \left( \ln(t-t_1) - \frac{M \ln(t-t_1)}{3C_0(t-t_1)} - \frac{f}{(t-t_1)} - \frac{5}{6(t-t_1)} \right) \right]
 \end{aligned}$$

where  $\rightarrow$  means asymptotically i. e. the ratio of the two sides approaches one as  $(t-t_1) \rightarrow \infty$ . Clearly by adjusting a constant,  $f$  can be made zero at any time.

$$J. \underline{\epsilon = \pm 1; a = \frac{1}{3}; \dot{\phi}_0 \neq 0}$$

For  $a = \frac{1}{3}$ ,  $\dot{\phi}_0 \neq 0$ , again only an expansion of 43) is obtained.

$$K. \underline{\epsilon = \pm 1; a = \frac{1}{2}; \dot{\phi}_0 = 0}$$

For  $a = \frac{1}{2}$ ,  $\dot{\phi}_0 = 0$ ,  $b = -\frac{2}{3}$  and 26) to order  $\omega^0$  yields

$$89) \quad R_0 = \left(\frac{3M}{4C_0}\right)^{\frac{1}{3}} (t-t_1)^{\frac{2}{3}}$$

and from 29)

$$90) \quad \dot{\phi}_1 = \frac{-M(t-t_0)}{2R_0^3}$$

or

$$91) \quad \phi_1 = -\frac{2}{3}C_0 \left[ \ln(t-t_1) - \left( \frac{t_1-t_0}{t-t_1} \right) \right]$$

26) to order  $\omega^{-1}$  then gives

$$92) \quad \frac{R_1}{R_0} = \frac{2}{9} \ln(t-t_1) + \frac{1}{18} + \frac{t_1-t_0}{18(t-t_1)^2} - \frac{9\epsilon}{20} \left(\frac{4C_0}{3M}\right)^{\frac{2}{3}} (t-t_1)^{\frac{2}{3}}$$

and

$$93) \quad \begin{aligned} \frac{\phi_2}{C_0} = & \frac{4}{3} \phi_1 + \frac{2}{9} \left[ \ln(t-t_1) \right]^2 - \frac{(t_1-t_0)^2}{18(t-t_1)^2} \\ & - \frac{27\epsilon}{20} \left( \frac{4C_0(t-t_1)}{3M} \right)^{\frac{2}{3}} - \frac{4}{9} (t_1-t_0) \left[ \frac{\ln(t-t_1)-1}{t-t_1} \right] \\ & - \frac{1}{54} \frac{(t_1-t_0)^2}{(t-t_1)^3} + \frac{27\epsilon}{20} \left( \frac{4C_0}{3M} \right)^{\frac{2}{3}} (t-t_1)^{-\frac{1}{3}} \end{aligned}$$

Thus

$$R_H = \frac{\omega}{3} (t-t_1) \left\{ 1 - \frac{1}{\omega} \left[ \frac{1}{3} - \frac{1}{2} \left( \frac{t_1-t_0}{t-t_1} \right)^2 - \frac{9\epsilon}{20} \left( \frac{4C_0(t-t_1)}{3M} \right)^{\frac{2}{3}} \right] + \dots \right\}$$

$$\rho = \frac{\omega^{\frac{3}{2}} 4C_0}{3(t-t_1)^2} \left\{ 1 - \frac{1}{\omega} \left[ \frac{2}{3} \ln(t-t_1) + \frac{1}{6} + \frac{(t_1-t_0)^2}{6(t-t_1)} - \frac{27\epsilon}{20} \left( \frac{4C_0(t-t_1)}{3M} \right)^{\frac{2}{3}} \right] + \dots \right\}$$

94)

$$M_H = 6\pi C_0 (t-t_1) \left\{ 1 - \frac{1}{\omega} \left[ \frac{2}{3} \ln(t-t_1) + \frac{7}{6} - \frac{1}{3} \left( \frac{t_1-t_0}{t-t_1} \right)^2 - \frac{27\epsilon}{10} \left( \frac{4C_0(t-t_1)}{3M} \right)^{\frac{2}{3}} \right] + \dots \right\}$$

$$D_H = 4\pi \left\{ 1 - \frac{1}{\omega} \left[ \frac{5}{6} + \frac{2}{3} \left( \frac{t_1-t_0}{t-t_1} \right) - \frac{1}{6} \left( \frac{t_1-t_0}{t-t_1} \right)^2 - \frac{9\epsilon}{4} \left( \frac{4C_0(t-t_1)}{3M} \right)^{\frac{2}{3}} \right] + \dots \right\}$$

Again the procedure diverges for small  $t-t_1$ , so that no information corresponding to the visible mass and radius is obtained from it.

$$L. \quad \underline{\epsilon = \pm 1; a = -\frac{1}{2}; \dot{\phi}_0 \neq 0}$$

For  $a = -\frac{1}{2}$ ,  $\dot{\phi}_0 \neq 0$ , 26), 29) reduce to

$$95) \quad \frac{3\epsilon}{\left(\frac{M}{2}\right)^{\frac{2}{3}}} \left( \frac{\dot{\phi}_0}{t-t_0} \right)^{\frac{2}{3}} = -\frac{2\dot{\phi}_0}{(t-t_0)\phi_0} - \frac{1}{2} \left( \frac{\dot{\phi}_0}{\phi_0} \right)^2$$

so that

$$96) \quad \phi_0 = (a + b(t-t_0)^2)^{-2}$$

with

$$97) \quad \frac{3\epsilon}{\left(\frac{M}{2}\right)^{\frac{2}{3}}} (4b)^{\frac{2}{3}} = 8ba$$

which is again just an expansion of 43).

$$M. \underline{\epsilon = \pm 1; a = -\frac{1}{2}; \dot{\phi}_0 = 0}$$

Finally,  $a = -\frac{1}{2}, \dot{\phi}_0 = 0$  is incompatible with  $\epsilon \neq 0$ .

N. Summary and Conclusions;  
Relation to Dirac's Conjecture

In summary, then, the following types of time dependence appear in  $\phi$  linearly:

$$t^{-4} \quad (\text{pages 105, 106})$$

$$t^{-\frac{2}{3\omega+4}} \quad (\text{pages 105, 106})$$

$$(a+t^2)^{-2} \quad (\text{page 107})$$

$$\ln t, (\ln t)^2 \quad (\text{page 114})$$

$$\frac{1}{t}, \frac{1}{t^2}, \frac{1}{t^3}$$

For reasonable ranges of  $\omega$ , none of these appear to resemble Dirac's  $\phi \sim t$  conjecture. Further to fit present values, very large ( $10^{40}$  or bigger in atomic units) constants must be chosen.

Of course, the assumption of a homogeneous universe without pressure cannot be regarded as realistic near the time origin. The behavior of the metric and scalar fields near this point where matter is in a highly condensed state would be very complex and probably bear little resemblance to that in the pressureless uniform model. It is possible that relatively simple conditions on  $\phi$  near the origin would then produce the large constant needed.

It seems that some sort of boundary condition will be needed to eliminate this extra constant (XIV-C page 104, XIII-A page 87) Perhaps spatial uniformity might be discarded and a time dependent model used in which a spatial condition, such as  $\phi \rightarrow 0$  outside matter, be imposed.

At any rate, it is clear that the present understanding of cosmological models is inadequate and will require further study. The main conclusion to be drawn from the study above is that a Friedmann type universe does not seem to predict a relation of the form  $\phi = C_0 t$ ;  $C_0 \sim 1$  over any significant interval of time.

O. Lowest Order Effects  
of Slowly Varying  $K$  on Planetary Orbits

In order to look at some of the effects of a slowly varying gravitational constant on Newtonian orbits, consider the particle Lagrangian,

$$98) \quad L = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r}$$

with  $k = k_0(1 + \delta t)$ ;  $k_0, \delta$  constant and  $\delta$  small. "Angular momentum"

$$99) \quad l \equiv r^2 \dot{\theta}$$

is conserved as usual because of rotational invariance. The "total energy,"  $E$ , however is not, and in fact,  $\frac{d}{dt}$  denoting total derivative along the path,

$$100) \quad \frac{dE}{dt} = - \frac{\dot{k}}{r} = - \frac{k_0 \delta}{r}$$

Hence the change per period is

$$\begin{aligned}
 \Delta E &= -k_0 \delta \int_0^{2\pi} \frac{d\theta}{r} \\
 101) \quad &= -\frac{k_0 \delta}{e} \int_0^{2\pi} r^2 d\theta \\
 &= \frac{2\pi k_0 \delta}{\sqrt{-2E}} + \text{order } \delta^2
 \end{aligned}$$

The change per period of the eccentricity  $e = \sqrt{1 + \frac{2E\ell^2}{k^2}}$  is thus

$$\begin{aligned}
 \Delta e &= \frac{\partial \ell^2}{\partial E} \left( \frac{\Delta E}{k^2} - \frac{\partial E}{\partial k^2} \Delta k \right) \\
 102) \quad &= 0 + \text{order } \delta^2
 \end{aligned}$$

## XV. Conservation Laws

### A. Introduction

Methods have already been given (IV-E, pages 21 f., IX-C, page 71 f.) for obtaining a "mass" associated with certain solutions to the field equations. More precisely, ways were discussed for obtaining experimentally measurable numbers from constants appearing in the field variables.

Other possible numbers can be obtained from conservation laws<sup>57</sup>. These will give constants associated with certain types of solutions which might be called "total masses" or "total energies." Of course these "masses" are less precisely defined in the sense that they are not as directly related to experimentally measured numbers as are the inertial and active gravitational masses discussed above.

In this section the structure of conservation laws associated

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57 For an extensive discussion of conservation laws and their uses see J. Fletcher, Rev. Mod. Phys. 32 65 (1960).

with the field equations VII-7), 8), 9) page 51 will be studied. . . In this case there are really two possible approaches. In the first, the field equations are divided by  $\phi$  to give an Einstein-type equation with modified energy tensor. Procedures used in the Einstein case are then applicable. Hence the expression of a conserved quantity as a function of the metric only is the same as in general relativity. The resultant conserved total "mass" has units of length, however, and corresponds to some averaged gravitational constant times total mass.

Alternate procedures yield a conserved "mass" having true units of mass and to which  $\int_{\text{matter}} \eta_{\alpha}^{\beta}$  instead of  $\phi^{-1} \int_{\text{matter}} \eta_{\alpha}^{\beta}$  contributes directly. This "total mass" is found to be equal to the "total Schwarzschild radius" times the asymptotic value of  $\phi$ . Further, both are linearly proportional to the inertial mass at least through first order in mass. Hence, the not too surprising result is obtained that for an isolated system inertial mass is at least approximately conserved as well as some sort of "total Schwarzschild radius."

#### B. Møller's Procedure: Evaluation of Constants

The most straightforward procedure for obtaining conservation laws is based on the assumption  $\phi \neq 0$  everywhere. The resulting equations may then be regarded as Einstein equations with modified matter tensor constructed from  $\phi$  and the variables describing matter itself. Hence the procedure described by Møller may be used, and the quantity  $\int_{\mathcal{V}} \gamma^{\nu}$ , defined by

$$1) \quad \int_{\mathcal{V}} \gamma^{\nu} = \int_{\mathcal{V}} \rho_{\mu}^{\nu} \gamma^{\mu}, \gamma^{\nu}$$

with

$$2) \quad P_{\mu}^{\nu} \equiv \sqrt{-g} \left[ -g^{\alpha\gamma} \left( -\Gamma_{\mu\alpha}^{\gamma} + \frac{1}{2} \delta_{\alpha}^{\gamma} \Gamma_{\mu\epsilon}^{\epsilon} + \delta_{\mu}^{\gamma} \Gamma_{\alpha\epsilon}^{\epsilon} \right) + \frac{1}{2} \delta_{\mu}^{\nu} g^{\alpha\beta} \left( -\Gamma_{\alpha\beta}^{\gamma} + \frac{1}{2} \delta_{\alpha}^{\gamma} \Gamma_{\beta\epsilon}^{\epsilon} + \frac{1}{2} \delta_{\beta}^{\gamma} \Gamma_{\alpha\epsilon}^{\epsilon} \right) \right]$$

satisfies

$$3) \quad \int_{\mu}^{\nu} \nu = 0$$

and the quantity

$$4) \quad P_{\mu} \equiv \int d^3x \int_{\mu}^0$$

can be considered as being proportional to the "total momentum." In particular for the static, spherically symmetric solution XII-19)-21), page 82 with constants B, C such that the metric becomes asymptotically flat and  $\phi$  approaches a constant,  $\frac{1}{k}$ , then

$$5) \quad P_i = 0 \quad \text{for } i \neq 0$$

and

$$6) \quad P_0 = - \frac{32\pi B}{\lambda} (C+1)$$

For the weak field case, this becomes to lowest order

$$7) \quad P_0 \approx - \left( \frac{2\omega-2}{2\omega-3} \right) K m$$

where

$$8) \quad m = \int d^3x T_{m0}^0$$

This is also the inertial mass, (IV-E, pages 21 f., X-C, pages 71 f.) at least approximately.

The "gravitational mass"  ${}_G P_0$  as measured by coordinate acceleration multiplied by  $4\pi$  times the square of coordinate distance at infinity is  ${}_G P_0 = \frac{32\pi B}{\lambda}$ . Hence

$$9) \quad \frac{P_0}{{}_G P_0} = C + 1$$

While from XII-23) page 82 to lowest order,  $C + 1 = \frac{\omega - 1}{\omega - 2}$ , independent of the inertial mass  $\mu$ , of the gravitating body, it might be thought that  $\mu$  would enter in higher order. That this is not true for next lowest order is seen from XII-11) page 80 giving C as the ratio of  $\rho_{nk} \phi$  to  $\alpha$ . For a single point particle of inertial mass  $\mu$  at rest at the origin, from IX-12) page 72,

$$\phi = \frac{1}{k} (1 + {}_2\psi + {}_4\psi + \dots) ; \quad h_{00} = -f - \frac{f^2}{2} + \dots$$

$$10) \quad {}_2\psi = \frac{f}{2\omega - 4} ; \quad h_{ij} = -\delta_{ij} \left( \frac{\omega - 1}{\omega - 2} \right) f + \dots ; \quad h_{0i} = 0 + \dots$$

$$f = - \frac{k_0 \mu}{4\pi \lambda}$$

and  ${}_4\psi$  satisfies

$$11) \quad {}_4\psi_{,i,i} + \left[ \frac{(-{}_2h_{00} + {}_2h_{i,i})}{{}_2} \psi_{,i} \right]_{,i} + ({}_2h^{i,i} \psi_{,i})_{,i} = 0$$

Thus,  ${}_4\psi \xrightarrow{\lambda \rightarrow \infty} 0$  yields

$$12) \quad {}_4\psi = \frac{f^2}{2(2\omega - 4)^2} ; \quad C = \frac{\rho_{nk} \phi}{\alpha} = \frac{1}{\omega - 2}$$

through first order in  $\mu$  and to this approximation "gravitational mass" and total energy defined in 6) above are proportional.

C. Criticism: Canonical Procedure

However, the main criticism that might be raised against this approach is that  $P_0$  so defined cannot meaningfully be compared with inertial mass since it is expressed in a unit of length and seems to correspond to some average gravitational constant times total energy. In fact, it is easily seen that the direct contribution of matter to  $\int \mathcal{T}^\nu$  is from a term of form  $\frac{\sqrt{-g} \mathcal{T}^\nu}{\phi}$ . There are several ways to eliminate this difficulty and construct a conserved quantity to which  $\sqrt{-g} \mathcal{T}^\nu$  contributes linearly. Perhaps the most straightforward is to proceed in standard fashion after obtaining a first order Lagrangian. Notice that

$$13) \quad \sqrt{-g} (\phi R + L + L_m) = \sqrt{-g} (\phi L_0 + L_1 + L_\phi + L_m) + \text{divergence}$$

with

$$14) \quad L_0 \equiv g^{\alpha\beta} \left( \Gamma_{\alpha\rho}^\sigma \Gamma_{\sigma\beta}^\rho - \Gamma_{\alpha\beta}^\rho \Gamma_{\rho\sigma}^\sigma \right)$$

$$L_1 \equiv g^{\alpha\beta} \left( \phi_{,\beta} \Gamma_{\alpha}^{\rho} - \phi_{,\rho} \Gamma_{\alpha\beta}^{\rho} \right)$$

Thus, with

$$15) \quad \sqrt{-g} \int_m \Gamma^{\alpha\beta} \delta g_{\alpha\beta} \equiv \delta (L_m \sqrt{-g})$$

$$16) \quad \left( \sqrt{-g} \int_m \Gamma^{\alpha\beta} \right)_{,\beta} - \frac{1}{\phi} \sqrt{-g} \int_m \Gamma^{\mu\nu} g_{\mu\nu,\alpha} = 0$$

and using the fact that the variation of the total Lagrangian is zero in the usual manner,

$$\begin{aligned}
 17) \quad 0 = \frac{\partial}{\partial x^\alpha} & \left[ -\sqrt{-g} g^{\mu\nu} \Gamma^\beta_{\mu\nu} + \frac{1}{\partial} \left( \delta_\alpha^\beta L_\phi - \phi_{,\alpha} \frac{\partial L_\phi}{\partial \phi_{,\beta}} \right) + \right. \\
 & + \frac{1}{\partial} \phi \left( \delta_\alpha^\beta L_0 - g_{\mu\nu, \alpha} \frac{\partial L_0}{\partial g_{\mu\nu, \beta}} \right) + \\
 & \left. + \frac{1}{\partial} \left( \delta_\alpha^\beta L_1 - g_{\mu\nu, \alpha} \frac{\partial L_1}{\partial g_{\mu\nu, \beta}} - \phi_{,\alpha} \frac{\partial L_1}{\partial \phi_{,\beta}} \right) \right]
 \end{aligned}$$

Although the conserved quantity in brackets can be expressed purely in terms of the metric and scalar fields, it does not seem to be the divergence of any reasonably simple functions.

#### D. Another Method: Summary

The second method is considerably more practical, however, namely, construct the conserved Einstein quantity as the divergence of an antisymmetric affine tensor. The latter can then be multiplied by  $\phi$ , and the divergence of the resulting quantity is again identically conserved. Specifically, following Pauli<sup>58</sup>, apart from sign changes, for  $R_{\mu}{}^{\nu\gamma}$  defined in 2)

$$18) \quad R_{\mu}{}^{\nu\gamma} = \mathcal{B}_{\mu}{}^{\nu\gamma} + \frac{1}{\partial} \delta_{\mu}{}^{\gamma} (\sqrt{-g} g^{\alpha\beta})_{,\alpha} - \frac{1}{\partial} (\sqrt{-g} g^{\nu\gamma})_{,\mu}$$

where

$$\begin{aligned}
 19) \quad \mathcal{B}_{\mu}{}^{\nu\gamma} = \frac{1}{\partial} \sqrt{-g} & \left\{ \delta_{\mu}{}^{\nu} (g^{\alpha\beta} \Gamma^{\gamma}_{\alpha\beta} - g^{\gamma\alpha} \Gamma^{\beta}_{\alpha\epsilon}) + \right. \\
 & \left. + \delta_{\mu}{}^{\gamma} (g^{\alpha\beta} \Gamma^{\nu}_{\alpha\epsilon} - g^{\nu\beta} \Gamma^{\alpha}_{\alpha\beta}) + g^{\gamma\alpha} \Gamma^{\nu}_{\alpha\mu} - g^{\nu\alpha} \Gamma^{\gamma}_{\alpha\mu} \right\} \\
 \mathcal{B}_{\mu}{}^{\nu\gamma} + \mathcal{B}_{\mu}{}^{\gamma\nu} & = 0 \\
 R_{\mu}{}^{\nu\gamma}{}_{,\gamma} & = \mathcal{B}_{\mu}{}^{\nu\gamma}{}_{,\gamma}
 \end{aligned}$$

58 W. Pauli, Theory of Relativity, (1958), pp. 215-216.

Hence, for the scalar theory define

$$20) \quad \bar{T}_\mu^\nu \equiv (\phi \partial_\mu \partial^\nu \phi)$$

so that

$$21) \quad \bar{T}_\mu^\nu{}_{,\nu} = 0$$

and  $\bar{T}_\mu^\nu$  is again an affine tensor.

Further, for the static case at any rate, it is clear that

$$22) \quad \int \bar{T}_0^0 d^3x = P_0 \lim_{\lambda \rightarrow \infty} \phi = K^{-1} P_0$$

with  $P_0$  defined in 4) above.

In summary then, a reasonable total energy, having dimensions of energy, can be constructed which is equal to the inertial mass,  $\mu$ , at least through second order in  $\mu$ .

APPENDIX A

If the space part of the metric XIV-5) page 101 is interpreted as being that of a unit  $\mathbb{S}^3$  sphere in a flat  $\mathbb{R}^4$  space, then it can be expressed

$$1) \quad dl^2 = (dx^2 + dy^2 + dz^2 + du^2) \Big|_{x^2 + y^2 + z^2 + u^2 = 1}$$

or in terms of polar coordinates  $\rho, \bar{\theta}, \bar{\phi}, \Psi$ , with

$$d\bar{\Omega}^2 = d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2$$

$$2) \quad dl^2 = \left[ d\rho^2 + \rho^2 (d\Psi^2 + \sin^2 \Psi d\bar{\Omega}^2) \right]_{\rho=1}$$

or

$$3) \quad dl^2 = d\Psi^2 + \sin^2 \Psi d\bar{\Omega}^2$$

Thus the coordinate system used in the calculation, in XIV,  $\Delta$ ,  $\theta, \phi$ , is given locally by

$$4) \quad \frac{\Delta}{2} = \tan \frac{\Psi}{2} ; \quad \theta = \bar{\theta} ; \quad \phi = \bar{\phi}$$

Hence, setting  $\Psi = 0$  for  $r = 0$ , as  $\Psi$  goes from 0 to  $2\pi$  around the universe at constant time,  $r$  goes from 0 to  $\infty$  at  $\Psi = \pi$ , then from  $-\infty$  back to 0 at  $\Psi = 2\pi$ . Further, for a light ray along  $d\bar{\Omega} = 0$ ,

$$5) \quad \int \frac{d\Delta}{(1 + \frac{\Delta^2}{4})} = \int d\Psi = \int \frac{dt}{R(t)}$$

so that, using XIV-43) page 107,

$$6) \quad \Psi_2 = 2 \sqrt{\frac{2w-3}{3}} \left( \tan^{-1} \frac{t}{\sqrt{2}} + \frac{\pi}{2} \right) = 2 \tan^{-1} \frac{\Delta_2}{2}$$

and performing  $\int \rho \sqrt{-g} d^3x$  in  $\psi, \bar{\theta}, \bar{\phi}$  coordinates

$$7) \quad M_V = 2\pi M \left[ \psi_v - \sin \psi_v \cos \psi_v \right]$$

Hence, as long as the phase is properly interpreted, XIV-48) page 108 can be formally used even for negative  $r_v$ .

APPENDIX B

Writing the equation as

$$1) \quad \dot{y} - f(y, x, \epsilon) = 0 \quad \dot{y} \equiv \frac{dy}{dx}$$

with  $y$  and  $f$ ,  $n$ -column matrices it is desired to investigate solutions which do not necessarily reduce, for small  $\epsilon$ , to  $y_0$ , the known (i. e. Einstein) solution to 1) with  $\epsilon = 0$ . For this purpose set

$$2) \quad y = E z$$

with  $E$  a diagonal  $n \times n$  matrix non-singular for  $\epsilon \neq 0$  and with non-zero elements consisting of powers of  $\epsilon$ . Further assume that

$$3) \quad f(y, x, \epsilon) = P F(z, x, \epsilon)$$

where  $F$  is continuous in  $\epsilon$  at  $\epsilon = 0$  and  $P$  is a matrix of the same sort as  $E$  and such that  $E^{-1}P$  has elements of  $\delta_{ij} \epsilon^{m_i}$  with  $m_i$  a positive or negative integer or zero. Hence, for  $\epsilon \neq 0$ , 1) becomes equivalent to

$$4) \quad Q(z, x, \epsilon) \equiv \epsilon^{-N} \dot{z} - \epsilon^{-N} E^{-1} P F(z, x, \epsilon) = 0$$

where

$$5) \quad N = \min(m_i, 0)$$

Assume now that  $Q \in C^{m+1}$  for some  $m > 0$ . In this case then a perturbation procedure will yield an approximate solution of order  $m$ ,

i. e. a function  $z(x)$  such that

$$6) \quad \lim_{\epsilon \rightarrow 0} \frac{Q(z'(x), z(x), x, \epsilon)}{\epsilon^m} = 0$$

The perturbation procedure is defined as follows. Writing

$$7) \quad z = \sum_{l=0}^m \epsilon^l z^l$$

with  $z^l$  independent of  $\epsilon$ , using the fact that  $Q \in C^{m+1}$

$$8) \quad Q = \sum_{l=0}^m \frac{\epsilon^l}{l!} \left( \frac{D^l Q}{D\epsilon^l} \right) \Big|_{\epsilon=0} + \epsilon^{m+1} H$$

with

$$9) \quad \frac{D}{D\epsilon} \equiv \left( \frac{\partial z^l}{\partial \epsilon} \frac{\partial}{\partial z^l} + \frac{\partial z^l}{\partial \epsilon} \frac{\partial}{\partial z^l} + \frac{\partial}{\partial \epsilon} \right)$$

and  $H$  is bounded for small  $\epsilon$ . Hence a n.s.c. for 6) is that for each  $P \leq m$

$$10) \quad \left( \frac{D^P Q}{D\epsilon^P} \right) \Big|_{\epsilon=0} = 0$$

For the particular form 4) of  $Q$ , if no  $n_i < 0$ ,  $N = 0$  and 10) for  $P = 0$  gives

$$11) \quad \lim_{\epsilon \rightarrow 0} \epsilon^i \left( F^l(z, x, 0) \right) = 0$$

From standard theorems,  $z^l$  then has exactly  $n$  independent constants, a number which clearly is already the maximum possible in  $z$  itself. Hence assume that  $n_i < 0$  for at least  $i$  and in fact let

$$12) \quad \eta_i < 0 \quad \text{for } i = 1, \dots, \nu$$

$$\eta_i \geq 0 \quad \text{for } i = \nu + 1, \dots, n$$

so that, for  ${}_0z$

$$13) \quad F^i({}_0z, \kappa, 0) = 0 \quad i \leq \nu$$

$$14) \quad \left. \frac{d}{d\epsilon} F^j({}_0z, \kappa, 0) \right|_{\epsilon=0} = 0 \quad j > \nu$$

For cases of interest,

$$15) \quad \det \left( \frac{\partial F^i}{\partial {}_0z^k} \right) \neq 0 \quad \text{for } i, k \leq \nu$$

so that 13) can be solved for  ${}_0z^1, \dots, {}_0z^\nu$  as functions of  $\kappa$  and  ${}_0z^{\nu+1}, \dots, {}_0z^n$ . Inserting this into 14) gives a standard equation for  ${}_0z^{\nu+1}, \dots, {}_0z^n$ . Hence the total solution  ${}_0z^i$  has exactly  $n - \nu$  independent constants  ${}_0C^q$ . From 10) for  $p \geq 1$

$$16) \quad \frac{\partial F^i}{\partial {}_0z^d}({}_0z, \kappa, 0) {}_0z^d = G^i({}_0z^1, \dots, {}_0z^\nu, \kappa) \quad i \leq \nu$$

$$17) \quad \left. \frac{d}{d\epsilon} F^k({}_0z, \kappa, 0) \right|_{\epsilon=0} {}_0z^d = G^k({}_0z^1, \dots, {}_0z^\nu, \kappa) \quad k > \nu$$

From 15), 16) may be solved for  ${}_p z^1, \dots, {}_p z^\nu$  as linear functions of  ${}_p z^{\nu+1}, \dots, {}_p z^n$ , 17) then giving a linear non-homogeneous equation for  ${}_p z^{\nu+1}, \dots, {}_p z^n$ . Denoting by  $\bar{z}(C^1, \dots, C^q)$  a special solution to 16), 17)

(the dependence on  ${}_0C''''_{p-1}$  coming from the  ${}_0Z''''_{p-1}$  in  $G^i$ ) the most general solution to 16) and 17) can be written

$$18) \quad {}_pZ({}_0C''''_p) = {}_p\bar{Z}({}_0C''''_{p-1}) + {}_pC^\alpha \frac{\partial {}_0Z({}_0C)}{\partial {}_0C^\alpha}$$

The main assertion of this discussion is then that to order  $\epsilon^m$ , the constants,  ${}_1C''''_m$ , may be arbitrarily fixed and in particular as introduced in 18), may be set zero. More precisely, for any  ${}_0C''''_m$  there exists a  ${}_0k$  such that

$$19) \quad \lim_{\epsilon \rightarrow 0} \frac{{}_1Z - {}_0Z({}_0k)}{\epsilon^m} = 0$$

where

$$20) \quad {}_1Z \equiv {}_0Z({}_0C) + \dots + \epsilon^m \left( \bar{Z}({}_0C''''_{m-1}) + {}_mC^\alpha \frac{\partial {}_0Z({}_0C)}{\partial {}_0C^\alpha} \right)$$

$$21) \quad {}_0Z \equiv {}_0Z({}_0k) + \dots + \epsilon^m \left( \bar{Z}({}_0k, 0, \dots, 0) \right)$$

In fact define a sequence,  ${}_0k^0, \dots, {}_0k^m \equiv {}_0k$  as follows. Let  ${}_0k^0 = {}_0C$  and assume that  ${}_0k^j$  has been defined for  $0 \leq j \leq i < m$  such that

$$22) \quad \lim_{\epsilon \rightarrow 0} \frac{{}_1Z - {}_0Z({}_0k^i)}{\epsilon^i} = 0$$

so that from 20), 21), for  $0 \leq j \leq i$

$$23) \quad {}_j^0Z({}_0k^i) = {}_0\bar{Z}({}_0C''''_{j-1}) + {}_jC^\alpha \frac{\partial {}_0Z({}_0C)}{\partial {}_0C^\alpha}$$

Since both  ${}^0_0 Z(K^i)$  and  ${}^i_0 Z$  are  $m^{\text{th}}$  order solutions to 4), both  ${}^0_{i+1} Z(K^i)$  and

$$24) \quad {}^0_{i+1} \bar{Z}(C^1, \dots, C^i) + {}^{i+1} C^\alpha \frac{\partial {}^0_0 Z(C)}{\partial C^\alpha}$$

are solutions to 16), 17), for  $P = i + 1$ , with the arguments  ${}^0_0 Z \dots Z = {}^i_0 Z$  of  $G^i$  on the right hand sides identical for both. Hence their differences is a solution to the corresponding homogeneous equations and there exists a set  ${}^{i+1} \Delta$  such that

$$25) \quad {}^0_{i+1} Z(K^i) + {}^{i+1} \Delta \frac{\partial {}^0_0 Z(C)}{\partial C^\alpha} = {}^0_{i+1} \bar{Z}(C^1, \dots, C^i) + {}^{i+1} C^\alpha \frac{\partial {}^0_0 Z}{\partial C^\alpha}$$

Defining

$$26) \quad {}^0 K^{i+1, \alpha} = {}^0 K^{i, \alpha} + {}^{i+1} \Delta \epsilon^{i+1}$$

then gives

$$27) \quad \lim_{\epsilon \rightarrow 0} \frac{{}^0_0 Z - {}^0_0 Z(K^i)}{\epsilon^{i+1}} = 0$$

Hence, by induction,  ${}^0 K^m = {}^0 K$  is defined as required in 19).

In the applications in the text (XIV)  $y$  will be  $(R, \phi)$  with 1) given by XIV-26) and XIV-29) pages 104, 105,  $x = t$  and  $\epsilon = \frac{1}{\omega}$ .

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REFERENCES

1. W. Davidson, Monthly Notices Roy. Astron. Soc. 117, 212.
2. R. H. Dicke, Rev. Mod. Phys. 29, 355 (1957).
3. R. H. Dicke, Jour. Wash. Acad. Sci. 48, 213 (1958).
4. R. H. Dicke, Science 129, 621 (1959).
5. R. H. Dicke, Am. Jour. Phys. 28, 344 (1960).
6. P. A. M. Dirac, Proc. Roy. Soc. (London) A165, 199 (1938).
7. A. Einstein, The Meaning of Relativity, Princeton, 1955, Fifth Edition.
8. M. Fierz, Helv. Phys. Acta 29, 128 (1956).
9. J. G. Fletcher, Rev. Mod. Phys. 32, 65 (1960).
10. V. Fock, J. Phys. (U.S.S.R.) 1, 81 (1939).
11. L. Infeld, Rev. Mod. Phys. 29, 398 (1957).
12. P. Jordan, Schwerkraft und Weltall, Braunschweig (1955), Second Edition.
13. P. Jordan, Z. Phys. 157, 112 (1959).
14. K. Just, Z. Phys. 140, 524, 648 (1955).
15. K. Just, Z. Phys. 141, 592 (1955).
16. K. Just, Z. Phys. 144, 411, 648 (1956).
17. G. Ludwig and K. Just, Z. Phys. 143, 472 (1955).
18. A. Papapetrou, Proc. Roy. Soc. (London) 64A, 57 (1951).
19. W. Pauli, Theory of Relativity, Pergamon Press, New York (1958).